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PSEUDO–RYAN REAL HYPERSURFACES OF A COMPLEX SPACE FORM *

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Introduction

A complex n–dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. The complete and simply connected complex space form consists of a complex projective space CP^n, a complex Euclidean space C^n or a complex hyperbolic space CH^n, according as $c > 0$, $c = 0$ or $c < 0$.

The induced almost contact metric structure and the Ricci tensor of a real hypersurface in $M_n(c)$ are respectively denoted by $\{\phi, <,>, \xi, \eta\}$ and S.

The study of real hypersurfaces of CP^n was initiated by Takagi [13], who proved that all homogeneous hypersurfaces of CP^n could be divided into six types which are said to be of type A_1, A_2, B, C, D and E. Moreover, he showed that if a real hypersurface M of CP^n has two or three distinct constant principal curvatures, then M is locally congruent to one of the homogeneous ones of type A_1, A_2 and B ([14]).

Recently, a characterization of the class of hypersurfaces with more than three distinct principal curvatures of CP^n is studied by Kimura [5], who proves the following interesting result :

Theorem K. *Let M be a real hypersurface of $CP^n (n \geq 3)$, then M satisfies $S\phi = \phi S$ if and only if M lies on a tube of radius r over one of the following Kaehlerian submanifolds :*

(A_1) *a hyperplane CP^{n-1},*

(A_2) *a totally geodesic $CP^k, (1 < k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*

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(B) *a copmlex quadric* Q^{n-1}, *where* $0 < r < \frac{\pi}{4}$ *and* $\cot^2 2r = n-2$,
(C) $CP^1 \times CP^{(n-1)/2}$, *where* $0 < r < \pi/4, \cot^2 2r = 1/(n-2)$ *and* $(n \geq 5)$ *is odd,*
(D) *a complex Grassmann* $G_{2,5}(C)$, *where* $0 < r < \frac{\pi}{4}, \cot^2 2r = 3/5$ *and* $n = 9$,
(E) *a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \pi/4$, $\cot^2 2r = 5/9$ *and* $n = 15$.

On the other hand, real hypersurfaces of CH^n have also been investigated by many authors (Berndt [1], Ki, Nakagawa and Suh [3], Ki and Suh [4], Montiel [9], Montiel and Romero [10] and Suh [12]).

Using some results about focal sets, Berndt [1] proved the following :

Theorem B. *Let* M *be a connected real hypersurface of* $CH^n (n \geq 2)$. *Then* M *has constant principal curvatures and* ξ *is principal if and only if* M *is locally congruent to one of the following:*
(A_0) *a horosphere in* CH^n,
(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* CH^{n-1},
(A_2) *a tube over a totally geodesic submanifold* CH^k *for* $k = 1, 2, \cdots, n-2$,
(B) *a tube over a totally real hyperbolic space* RH^n.

It is necessary to remark that real hypersurfaces of type A_0 or A_1 appearing in Theorem B, are totally η-umblical hypersurfaces with two distinct constant principal curvatures. In the paper of Montiel [9] the real hypersurfaces of type A_0 in Theorem B is said to be self-tube.

In particular, it is proved in [4] that a real hypersurface of $CH^n (n \geq 3)$ satisfies $S\phi = \phi S$ if and only if M is of type A_0, A_1, or A_2.

We now introduce the notion of a pseudo-Ryan real hypersurface in $M_n(c)$, which is defined by $< R(Z,W)SX, Y >= 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ, where R is denoted by the Riemannian curvature tensor of M. The main purpose of the present paper is to investigate pseudo-Ryan real hypersurfaces of $M_n(c), c \neq 0$ by using above classification theorems.

1. Preliminaries

Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ and let C be a unit normal vector field on a neighborhood of a point $x \in M$. We denote by J the Kaehlerian structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformation of X and C under J can be represented by

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defined a skew–symmetric transformation on the tangent bundle of M, η and ξ being denoted by a 1–form and a vector field on a neighborhood of x in M respectively. Denoting $<,>$ by the induced Riemannian metric on M, it is seen that $< \xi, X >= \eta(X)$ for any tangent vector X on M. By the properties of the almost complex structure J, we see that

$$\phi^2 = -I + \eta \otimes \xi, \phi\xi = 0, \eta \circ \phi = 0, \eta(\xi) = 1,$$

where I denotes the identity transformation, the aggregate $(\phi, <,>, \xi, \eta)$ is called an almost contact metric structure. Furthermore, the covariant derivatives of the structure tensors are given by

$$(1.1) \qquad (\nabla_X \phi)Y = - < AX, Y > \xi + \eta(Y)AX, \quad \nabla_X \xi = \phi AX,$$

where ∇ is the induced Riemannian connection on M and A denotes the shape operator in the direction of C. The tangent space of M at x will be denoted by $T_x(M)$.

In the sequel, the ambient Kaehlerian manifold is assumed to be of constant holomorphic sectional curvature c, which is called a complex space form and denoted by $M_n(c)$. Then the equations of Gauss and Codazzi are respectively obtained:

$$\begin{aligned} < R(X,Y)Z, W > \; = \; & \frac{c}{4}\{< Y, Z >< X, W > - < X, Z >< Y, W > \\ (1.2) \qquad & + < \phi Y, Z >< \phi X, W > \\ & - < \phi X, Z >< \phi Y, W > -2 < \phi X, Y >< \phi Z, W >\} \\ & + < AY, Z >< AX, W > - < AX, Z >< AY, W >, \end{aligned}$$

$$(1.3) \; (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)Y - \eta(Y)\phi X - 2 < \phi X, Y > \xi\},$$

where R denotes the Riemannian curvature tensor. The Ricci tensor S' of M is the tensor field of type $(0,2)$ given by $S'(X,Y) = Tr\{Z \to R(Z,X)Y\}$. Also, it may be regarded as the tensor field S of type (1,1) defined by $S'(X,Y) = < SX, Y >$. Thus, by means of (1.2) the Ricci tensor S of M is given by

$$S = \frac{c}{4}\{(2n+1)I - 3\eta \otimes \xi\} - P, \tag{1.4}$$

where $P = A^2 - hA$ and $h = TrA$.

The structure vector ξ is principal, namely, if

$$A\xi = \alpha\xi, \tag{1.5}$$

where α is the principal curvature corresponding ξ. In this case, it is known that α is a locally constant on M (see [4] and [8]). The covariant derivative gives

$$(\nabla_X A)\xi = \alpha\phi AX - A\phi AX,$$

where we have used the second formula of (1.1), which together with the equation of Codazzi (1.3) yields

$$A\phi AX = \frac{\alpha}{2}(\phi A + A\phi)X + \frac{c}{4}\phi X. \tag{1.6}$$

2. Pseudo–Ryan real hypersurfaces

Let M be a real hypersurfaces of a complex space form $M_n(c), c \neq 0$, $n \geq 3$. The real hypersurface M is said to be *pseudo–Ryan* if $RS = f\xi$ for any function f on M, that is, $< R(Z,W)S(X), Y >= 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ. Then by the properties of the Riemannian curvature tensor, we obtain

$$< R(Z,W)S(X), Y > + < R(Z,W)S(Y), X >= 0,$$

which together with (1.4) gives

$$< R(Z,W)(PX), Y > + < R(Z,W)(PY), X >= 0 \tag{2.1}$$

for any X, Y, Z and W in $\xi^\perp$, where $\xi^\perp$ denotes the orthogonal complement of ξ in $T_x(M)$ for any x in M.

Let X be a principal curvature vector of A orthogonal to ξ with principal curvature λ. Then, by the definition of P, we have

$$PX = \alpha_1 X, \tag{2.2}$$

where we have put $\alpha_1 = \lambda^2 - h\lambda$. Similary, for $AY = \mu Y$ we have

$$PY = \alpha_2 Y, \quad \alpha_2 = \mu^2 - h\mu. \tag{2.3}$$

Accordingly, (2.1) turns out to be $(\alpha_1 - \alpha_2) < R(Z,W)X, Y >= 0$ and hence

$$(\lambda - \mu)(\lambda + \mu - h) < R(Z,W)X, Y >= 0 \tag{2.4}$$

for any Z and W in $\xi^\perp$.

If we put $X = W$ and $Z = Y = \phi X$, then (1.2) is reduced to

$$\begin{aligned} < R(X,\phi X)(\phi X), X > &= c+ < A\phi X, X >< AX, X > \\ &\quad - < AX, \phi X >< A\phi X, X > \end{aligned}$$

for a unit tangent vector field X orthogonal to ξ. Combining the last two equations and making use of (2.2) and (2.4), we get

$$(\lambda - \mu)(\lambda + \mu - h)(c + \lambda\mu) = 0. \tag{2.5}$$

On the other hand, for a unit tangent vector Z orthogonal to ξ, the Gauss equation (1.2) implies

$$\begin{aligned} < R(\phi Z, Z)X, Y > &= \frac{c}{2}\{< Z, \phi X >< Z, Y > - < Z, \phi Y >< Z, X > \\ &\quad + < \phi X, Y >\} - < Z, \phi AY >< Z, AX > \\ &\quad + < Z, \phi AX >< Z, AY > . \end{aligned}$$

Now, we take an orthogonal frame $\{E_1, \cdots, E_{2n-2}, \xi\}$ of $T_x(M)$. Then the last relationship leads to

$$\sum_{i=1}^{2n-2} < R(\phi E_i, E_i)X, Y >= nc < \phi X, Y > -2 < \phi AY, AX > .$$

Let $AX = \lambda X$ and $AY = \mu Y$ for X and $Y \in \xi^\perp$. We then have

$$\sum_{i=1}^{2n-2} < R(\phi E_i, E_i)X, Y >= (nc + 2\lambda\mu) < \phi X, Y >,$$

which together with (2.4) yields $(\lambda - \mu)(\lambda + \mu - h)(nc + 2\lambda\mu) = 0$. From this fact and (2.5), it follows that

$$(\lambda - \mu)(\lambda + \mu - h) = 0, \tag{2.6}$$

which is equivalent to $\alpha_1 = \alpha_2$ because of (2.2) and (2.3).

In what follows, we denote by $P(\alpha_r)$ and $A(\lambda)$ the eigenspace of P and A for each point x in M associated with eigenvalue α_r and λ respectively. Summing up, we have

Lemma 1. *Let M be a pseudo-Ryan real hypersurface of $M_n(c), c \neq 0, n \geq 3$. If $AX = \lambda X$ and $A\phi X = \mu\phi X$ for any $X \in P(\alpha_1)$, then we have $\phi X \in P(\alpha_1)$ and (2.6).*

Since P can be regarded as the symmetric linear transformation of $T_x(M)$ for each x in M, the orthogonal complement $\xi^\perp$ can be decomposed as follows :

$$\xi^\perp = P(\alpha_1) \oplus P(\alpha_2) \oplus \cdots \oplus P(\alpha_p),$$

where $\alpha_1, \cdots, \alpha_p$ are mutually distinct at x in M.

For unit vectors $X \in A(\lambda)$ and $Z \in A(\sigma)$ such that X, Z and $\phi X (\in \xi^\perp)$ is orthonomal, we can easily, using (2.1), see that

$$(\lambda - \sigma)(\lambda + \sigma - h) < R(X,Y)Z, W >= 0,$$

from which, by putting $X = W$ and $Y = Z$, we have

$$(\lambda - \sigma)(\lambda + \sigma - h)(\frac{c}{4} + \lambda\sigma) = 0. \tag{2.7}$$

In the same way, for $Y \in A(\mu)$ and $Z \in A(\sigma)$ we obtain

$$(\mu - \sigma)(\mu + \sigma - h)(\frac{c}{4} + \mu\sigma) = 0. \tag{2.8}$$

3. A characterization of pseudo–Ryan real hypersurfaces

Let M be a pseudo–Ryan real hypersurface of $M_n(c), c \neq 0$, such that ξ is principal. We then easily, taking account of (1.6), see that

$$(2\lambda - \alpha)A\phi X = (\alpha\lambda + \frac{c}{2})\phi X \tag{3.1}$$

for a unit vector field $X \in A(\lambda)$.

Lemma 2. *Let M be a pseudo-Ryan real hypersurface of $M_n(c), c \neq 0, n \geq 3$, on which ξ is principal. Then the number of distinct eigenvalues of P is one, that is, $p = 1$.*

Proof. Suppose that $p \geq 2$, or any $Y \in A(\sigma) \subset P(\alpha_1)$, we then have $\alpha_r = \sigma^2 - h\sigma$, $r \geq 2$ because of (2.3) and hence it follows that $\alpha_1 \neq \alpha_2$, namely, $(\lambda - \sigma)(\lambda + \sigma - h) \neq 0$. Consequently (2.7) implies $\lambda\sigma + \frac{c}{4} = 0$ because $\{X, \phi X, Y\}$ which we have taken orthonormal.

Similarly we have from (2.8) $\mu\sigma + \frac{c}{4} = 0$. Thus, last two relationships tell us that $(\lambda - \mu)\sigma = 0$. However, σ can not be zero because of $c \neq 0$. Thus, we have $\lambda = \mu$, which together with (3.1) implies that

$$\lambda^2 - \alpha\lambda - \frac{c}{4} = 0. \tag{3.2}$$

Since λ satisfies the quadratic equation $x^2 - \alpha x - \frac{c}{4} = 0$ with constant coefficients, it follows that $\lambda = \frac{1}{2}(\alpha \pm \sqrt{D}), D = \alpha^2 + c \geq 0$. But, it is seen that $D > 0$. In fact, if $D = 0$, then $\lambda = \frac{\alpha}{2}$. Therefore, all principal curvatures of M are $\alpha, \frac{\alpha}{2}, \cdots, \frac{\alpha}{2}$. It means that $\xi^{\perp} = A(\lambda) = P(\alpha_1)$, which implies $p = 1$. It is contradictory. Hence, the quadratic equation $x^2 - \alpha x - \frac{c}{4} = 0$ has following solutions :

$$c > 0; \frac{\sqrt{c}}{2} \cot\theta \text{ or } \frac{-\sqrt{c}}{2} \tan\theta, (0 < \theta < \frac{\pi}{2})$$

$$c < 0; \frac{\sqrt{-c}}{2} \coth\theta \text{ or } \frac{\sqrt{-c}}{2} \tanh\theta, (\theta \neq 0).$$

Since we have $\sigma = -\frac{c}{4\lambda}$, it follows that σ is the same as above and thus $\alpha_1 = \lambda^2 - h\lambda = \alpha_r$, which produces a contradiction. Thus, we arrive at $p = 1$. This completes the proof of Lemma 2.

By Lemma 2, it is seen that $\xi^{\perp} = P(\alpha_1)$ and $\dim \xi^{\perp} = 2n - 2$ for $n \geq 3$. Hence, it follows that $\xi = A(\lambda) \oplus A(\mu)$ or $A(\lambda)$.

For the case where $P(\alpha_1) = A(\lambda)$, we get $\lambda = \mu$ and hence λ is a root of the quadratic equation $x^2 - \alpha x - \frac{c}{4} = 0$, which means that λ is constant.

The case where $P(\alpha_1) = A(\lambda) \oplus A(\mu)$, $(\lambda \neq \mu)$ is considered. In this case, we obtain

$$\lambda + \mu = h \tag{3.3}$$

because of (2.6). It is not hard to see that $Y \in A(\lambda) \to \phi Y \in A(\mu)$ is injective because of $\lambda \neq \mu$. Since it is known that $\dim A(\lambda) = \dim A(\mu) = n - 1$ ([9], [14]), it follows that $h = \alpha + (n-1)(\lambda + \mu)$, which together with (3.3) yields $\alpha + (n-2)h = 0$. Accordingly, we see that $h =$ constant and thus λ and μ are constant. Therefore, we verify, in any case, that all principal curvatures of M are constant.

Lemma 3. *Under tha same assumptions as that in Lemma 2, we have*

$$P\phi = \phi P \ i.e. \ S\phi = \phi S.$$

Proof. Let $Q = P\phi - \phi P$. Then we have $Q\xi = 0$ because ξ is principal and $P\phi\xi = \phi P\xi$. For any $X \in \xi^{\perp} = P(\alpha_1)$, it is, using Lemma 1, seen that $\phi X \in P(\alpha_1)$. Consequently we obtain $PX = \alpha_1 X$ and $P\phi X = \alpha_1 \phi X$ and thus $QX = 0$ for any $X \in \xi^{\perp}$. Therefore we have $P\phi = \phi P$. This completes the proof.

From Theorem 3.3 of [4] and Lemma 3, we see that $S\phi = \phi S$ if and only if M is of type A_0, A_1, A_2 when $c < 0$. Thus, for $\lambda \neq \mu$ we may only consider the case where $c > 0$. In this case we have (3.3). The table of Takagi [13] gives that

$$\alpha = \sqrt{c}\cot 2\theta, \lambda = \frac{\sqrt{c}}{2}\cot(\theta - \frac{\pi}{4}) \text{ and } \mu = -\frac{\sqrt{c}}{2}\tan(\theta - \frac{\pi}{4}).$$

Thus, it follows that $\lambda + \mu = \frac{-c}{\alpha}$, which together with (3.3) implies that $\alpha^2 = (n-2)c$, namely, $\cot^2 2\theta = n - 2$. From this fact, Lemma 3 and Theorem K, we see that $S\phi = \phi S$ if and only if M is of type A_1, A_2 or B when $c > 0$.

Summing up, we have

Theorem 4. *Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0$, $n \geq 3$. Then M is pseudo-Ryan and the structure vector ξ is principal if and only if M is locally congruent to one of the type A_1, A_2 or B when $c > 0; A_0, A_1$ or A_2 when $c < 0$.*

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