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ON MULTIPLE INTEGRAL RELATION AND ITS APPLICATION TO SPHEROIDAL FUNCTIONS

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1. Introduction

Let us abbreviate, for convenience, the p–parameter sequence $(a_1, \alpha_1), \cdots, (a_p, \alpha_p)$, and the q–parameter sequence $(b_1, \beta_1), \cdots, (b_q, \beta_q)$, by $(a_j, \alpha_j)_{1,p}$ and $(b_j, \beta_j)_{1,q}$ respectively.

We start by recalling the familiar H–function in the form (cf. [8], p. 408):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{matrix}\right.\right] \tag{1}$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j-\beta_j s)\prod_{j=1}^n \Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^p \Gamma(a_j-\alpha_j s)} \times z^s ds$$

where L is a suitable contour.

The H–function of two complex variables was defined and represented by Mittal and Gupta ([13], p. 117).

The H–function of several complex variables occurring in this paper has been defined and represented by Srivastava & Panda (cf. [17], p. 271) as follows:

$$H_{A,C:(B,D);\cdots;(B^{(r)},D^{(r)})}^{0,\lambda:(u',v');\cdots;(u^{(r)},v^{(r)})}\left[\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[(b'):\varphi'];\cdots;[(b^{(r)}:\varphi^{(r)}];\\ \\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'];\cdots;[(d^{(r)}:\delta^{(r)}];\end{matrix} z_1, z_2, \cdots, z_r\right]$$
$$= \frac{1}{(2\pi i)^r}\int_{L_1}\int_{L_2}\cdots\int_{L_r} \psi(s_1,\cdots,s_r)\prod_{k=1}^r[\varphi_k(s_k)z^{s_k}]ds_1\cdots ds_r \tag{2}$$

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where

$$\psi(s_1, s_2, \cdots, s_r) = \frac{\prod_{j=1}^{\lambda} \Gamma[1 - a_j + \sum_{k=1}^{r} \theta^{(k)} s_k]}{\prod_{j=\lambda+1}^{A} \Gamma[a_j - \sum_{k=1}^{r} \theta_j^{(k)} s_k] \prod_{j=1}^{C} \Gamma[1 - c_j + \sum_{k=1}^{r} \psi_j^{(k)} s_k]} \tag{3}$$

$$\varphi_k(s_k) = \frac{\prod_{j=1}^{u^{(k)}} \Gamma[d_j^{(k)} - \delta_j^{(k)} s_k] \prod_{j=1}^{v^{(k)}} \Gamma[1 - b_j^{(k)} + \varphi_j^{(k)} s_k]}{\prod_{j=u^{(k)}+1}^{D^{(k)}} \Gamma[1 - d_j^{(k)} + \delta_j^{(k)} s_k] \prod_{j=v^{(k)}+1}^{B^{(k)}} \Gamma[b_j^{(k)} - \varphi_j^{(k)} s_k]}, \quad (k = 1, \cdots, r) \tag{4}$$

and $L_1, L_2, \cdots, L_r$ are suitable contours.

The multiple integral (2) converges absolutely if

$$|arg(z)| < (1/2) \cdot \pi \cdot \Delta_k, \quad k = 1, \cdots, r \tag{5}$$

where

$$\Delta_k = \sum_{j=1}^{\lambda} \theta_j^{(k)} - \sum_{j=\lambda+1}^{A} \theta_j^{(k)} + \sum_{j=1}^{v^{(k)}} \varphi_j^{(k)} + \sum_{j=v^{(k)}+1}^{B^{(k)}} \varphi_j^{(k)} - \sum_{j=1}^{C} \psi_j^{(k)} + \sum_{j=1}^{u^{(k)}} \delta_j^{(k)} - \sum_{j=u^{(k)}+1}^{D^{(k)}} \delta_j^{(k)} > 0, (k = 1, \cdots, r) \tag{6}$$

wherever no confusion arises we write the first member of (2) in the form $H[z_1, \cdots, z_r]$ and $H^*[z_1, \cdots, z_r]$ when $\lambda = 0$.

The asymptotic expansion of $H[z_1, \cdots, z_r]$ is given by :

$$H[z_1, \cdots, z_r] = \begin{cases} 0(|z_1|^{\alpha_1} \cdots |z_r|^{\alpha_r}), \max\{|z_1|, \cdots, |z_r|\} \to 0 \\ 0(|z_1|^{-\beta_1} \cdots |z_r|^{-\beta_r}), \lambda \equiv \min\{|z_1|, \cdots, |z_r|\} \to \infty \end{cases} \tag{7}$$

where with $k = 1, \cdots, r$,

$$\alpha_k = d_j^{(k)} / \delta_j^{(k)}, \quad j = 1, \cdots, u^{(k)}$$
$$\beta_k = [1 - b_j^{(k)}] / \varphi_j^{(k)}, \quad j = 1, \cdots, v^{(k)} \tag{8}$$

The known results : ([14], eq. (14); [11], p. 226 (7) ; [9], p. 599; [1], p. 279; [5], p. 172; [10], p. 145; [6], p. 310, 6.2(19)) required in the sequel are given below :

(i) The spheroidal function $\psi_{\alpha_n}(c,\eta)$ of general order $\alpha > -1$ can be expanded as :

$$\psi_{\alpha_n}(c,\eta) = \frac{i^n\sqrt{2\pi}}{v_{\alpha^n}(c)}\sum *_{k=0,1}^{\infty} a_k(c|\alpha_n)(c\eta)^{-(\alpha+1/2)}J_{k+\alpha+1/2}(c\eta) \qquad (9)$$

which represents the function uniformly on $(-\infty,\infty)$, where the coefficients $a_k(c|\alpha_n)$ satisfy the recursion formula ([14], eq. (67)), and the asterisk $(*)$ over the summation sign indicates that the sum is taken over only even or odd values of k according as n is even or odd.

(ii)

$$z^u \cdot J_v(z) = z^u \cdot H_{0,2}^{1.0}\left[z^2/4\middle|_{((1/2)(u+v),1),((1/2)(u-v),1)}\right] \qquad (10)$$

(iii)

$$H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{matrix}\right] = c\cdot H_{p,q}^{m,n}\left[z^c\middle|\begin{matrix}(a_j,c\alpha_j)_{1,p}\\ \\ (b_j,c\beta_j)_{1,q}\end{matrix}\right], c>0 \qquad (11)$$

(iv)

$$(1+z)^{-\alpha} = \frac{1}{\Gamma(\alpha)}H_{1,1}^{1,1}\left[z\middle|\begin{matrix}(1-\alpha,1)\\ \\ (0,1)\end{matrix}\right] \qquad (12)$$

(v)

$$H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{matrix}\right] \qquad (13)$$
$$= \sum_{h=0}^{m}\sum_{t=0}^{\infty}\frac{(-1)^t z^{\eta_t}}{t!\beta_h}\frac{\prod_{j=1}^{m}\Gamma_{j\neq h}(b_j-\beta_j\eta_t)\prod_{j=1}^{n}\Gamma(1-a_j+\epsilon_j\eta_t)}{\prod_{j=n+1}^{p}\Gamma(a_j-\epsilon_j\eta_t)\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\eta_t)},$$

where $\eta_t = (b_h+t)/\beta_h$.

(vi)

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{(\prod_{k=1}^{r}(x_k^{\alpha_k^{-1}})f(\sum_{k=1}^{r}x_k^{\rho_k})}{\left[\sigma_0+\mathcal{T}(\sum_{k=1}^{r}x_k^{\sigma_k})^h\right]^{\sum_1^r(\alpha_k/\sigma_k)}}dx_1\cdots dx_r \qquad (14)$$
$$= \frac{1}{\prod_{k=1}^{r}(\rho_k)}\frac{\prod_{k=1}^{r}\Gamma(\alpha_k/\rho_k)}{\Gamma(\sum_{k=1}^{r}\alpha_k/\rho_k)}\int_0^\infty\frac{z^{\sum_{k=1}^{r}(\alpha_k/\rho_k)-1}f(z)}{(\sigma_0+\mathcal{T}z^h)^{\sum(\alpha_k/\rho_k)}},$$

where $\alpha_k,\rho_k,\mathcal{T},\sigma_0(k=1,\cdots,r)$ are all positive quantities.

(vii)

$$\int_0^\infty \frac{x^{\rho-1}}{(1+ax^h)^\mu} = \frac{a^{-\rho/h}}{h}\frac{\Gamma(\rho/h)\Gamma(\mu-\rho/h)}{\Gamma(\mu)} \tag{15}$$
$$|arg\, a| < \pi, \quad Re(\rho) < Re(\mu).$$

2. The Multiple Integral Relation

In this section, we establish the multiple integral relation:

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\prod_{k=1}^r (x_k^{\alpha_k^{-1}}) f(\sum_{k=1}^r x_k^{\sigma_k})}{\{\sigma_0 + \mathcal{T}(\sum_{k=1}^r x_k^{\sigma_k})^h\}^{\sum_1^r (\alpha_k/\sigma_k)}}$$
$$H_{p,q}^{m,n}\left[\frac{b\prod_{k=1}^r (x_k^{u_k})}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\sum_1^r (u_k/\sigma_k)}}\Bigg|_{(b_j,\beta_j)_{1,q}}^{(a_j,\epsilon_j)_{1,p}}\right] dx_1 dx_2 \cdots dx_r$$
$$= \prod_{k=1}^r (\sigma_k)^{-1} \int_0^\infty \frac{z^{\sum_{k=1}^r (\alpha_k/\sigma_k)^{-1}} \cdot f(z)}{(\sigma_0 + \mathcal{T} z^h)^{\sum_1^r (\alpha_k/\sigma_k)}} H_{p+r,q+1}^{m,n+r} \tag{16}$$
$$\left[\frac{b z^{\sum_1^r (u_k/\sigma_k)}}{(\sigma_0 + \mathcal{T} z^h)^{\sum_1^r (u_k/\sigma_k)}}\Bigg|_{(b_j,\beta_j)_{1,q}, \quad ,(1-\sum_1^r(\alpha_k/\sigma_k), \sum_1^r (u_k/\sigma_k))}^{(1-\alpha_1/\sigma_1, u_1/\sigma_1),\cdots,(1-\alpha_r/\sigma_r, u_r/\sigma_r),(a_j,\epsilon_j)_{1,p}}\right] dz$$

which is valid under the following set of conditions:

(i) $\alpha_k, \sigma_k, u_k (k = 1, \cdots, r)$ and $\mathcal{T}, h, \sigma_0$ are all positive quantities and b is a complex quantity.

(ii) $Re[\sum_{k=1}^r (\alpha_k/\sigma_k + (b_w/\beta_w)(u_k/\sigma_k)] > 0, (w = 1, \cdots, m)$

$$D = \sum_{j=1}^n (\epsilon_j) - \sum_{n+1}^P (\epsilon_j) + \sum_1^m (\beta_j) - \sum_{m+1}^q (\beta_j) > 0$$

$$|arg\, b| < (1/2) \cdot D \cdot \pi, \sum_1^q (\beta_j) - \sum_1^p (\epsilon_j) > 0.$$

Proof. Let

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\prod_1^r (x_k^{\alpha_k^{-1}}) f(\sum_{k=1}^r x_k^{\sigma_k})}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\sum_1^r (\alpha_k/\sigma_k)}}$$
$$H_{p,q}^{m,n}\left[\frac{b\prod_1^r (x_k^{u_k})}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\sum_1^r (u_k/\sigma_k)}}\Bigg|_{(b_j,\beta_j)_{1,q}}^{(a_j,\epsilon_j)_{1,p}}\right] dx_1 dx_2 \cdots dx_r = I \text{ (say)}$$

Now expressing H–function in its contour integral form (1), we get:

$$I = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\prod_1^r (x_k^{\alpha_k^{-1}}) f(\sum_1^r x_k^{\sigma_k})}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\sum_1^r (\alpha_k/\sigma_k)}} \left\{ \frac{1}{2\pi i} \int_L \theta(\xi) \right.$$
$$\left. \times \left[\frac{b(\prod_1^r x_k^{u_k})}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\sum_1^r (u_k/\sigma_k)}} \right]^\xi d\xi \right\} dx_1 dx_2 \cdots dx_r \tag{17}$$

Now on changing the order of integration on the right–side of (17) (which is permissible under the conditions stated), expressing the inner integral thus obtained by an appeal to the multiple integral relation (14), we get:

$$I = \frac{1}{2\pi i} \int_L^\xi b \cdot \theta(\xi) \left[\prod_1^r (\sigma_k^{-1}) \frac{\prod_1^r \Gamma[\frac{\alpha_k + u_k \xi}{\sigma_k}]}{\Gamma[\sum_1^r (\frac{\alpha_k + u_k \xi}{\sigma_k})]} \right. \tag{18}$$
$$\left. \times \int_0^\infty \frac{z^{[\sum_1^r (\alpha_k + u_k \xi)/\sigma_k] - 1} \cdot f(z)}{\prod_1^r [(\sigma_0 + \mathcal{T} z)^{(\alpha_k + u_k \xi)/\sigma_k}]} dz \right] d\xi$$

Again changing the order of integration in (18) and interpreting the ξ–integral by virtue of (1), we arrive at the result (16).

3. Application of Integral Relation (16)

As an application of integral relation (16) we obtain the following integral:

$$I = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\prod_1^r x_k^{\alpha_k^{-1}} (\sum_1^r x_k^{\sigma_k})^\rho}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\sum_1^k (\alpha_k/\sigma_k)}} \cdot \psi_{\alpha\nu} \left[c, \right.$$
$$\left. \frac{(\sum_1^r x_k^{\sigma_k})^{\ell_1}}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\ell_2}} \right] \cdot H_{p,q}^{m,n} \left[\frac{b \prod_1^r (x_k)^{u_k}}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\sum_1^r (u_k/\sigma_k)}} \middle|_{(b_j,\beta_j)_{1,q}}^{(a_j,\epsilon_j)_{1,p}} \right]$$
$$H \left[\frac{w_1 (\sum_1^r x_k^{\sigma_k})^{v_1}}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\rho_1}} \cdots, \frac{w_r (\sum_1^r x_k^{\sigma_k})^{v_r}}{\{\sigma_0 + \mathcal{T}(\sum_1^r x_k^{\sigma_k})^h\}^{\rho_r}} \right]$$
$$= \frac{i^\nu \cdot \sqrt{\pi} \prod_1^r (\sigma_k^{-1})}{v_{\alpha\nu}(c) 2^{(\alpha+1)} \cdot h} \frac{\mathcal{T}_1^{-(1/h)[\sum_1^r (\alpha_k/\sigma_k) + \rho]}}{\sigma_0^{\sum_1^r (\alpha_k/\sigma_k)}} \tag{19}$$
$$\cdot \sum_{h=1}^m \sum_{t=0}^\infty \sum {}^*{}_{k=0,1}^\infty a_k(c|\alpha\nu) \cdot \frac{\prod_1^m \Gamma(b_j - \eta_t \beta_j) \prod_1^n \Gamma(1 - a_j + \epsilon_j \eta_t)}{\prod_{m+1}^q \Gamma(1 - b_j + \eta_t \beta_j) \prod_{n+1}^p \Gamma(a_j - \epsilon_j \eta_t)}$$
$$\cdot \left[b(\sigma_0^h, \mathcal{T}_1)^{(-1/h) \sum_1^r (u_k/\sigma_k)} \right]^{\eta_t} \frac{(-1)^t}{t!} \cdot \frac{1}{\beta_\ell} \frac{\prod_1^r \Gamma(\alpha_k/\sigma_k + \eta_t (u_k/\sigma_k))}{\Gamma[\sum_1^r (\alpha_k/\sigma_k + \eta_t (u_k/\sigma_k))]}$$

$$H^{0,\lambda+2:(u',v'),\cdots,(u^{(r)},v^{(r)});(1,0)}_{A+2,C+1:(B',D');\cdots;(B^{(r)},D^{(r)});(0,2)}\left[\begin{matrix} R:[--];\cdots;[--];----------------; \\ S:[--];\cdots;[--]:(k/2,1/2,(-\alpha-k/2-1/2,1/2); \end{matrix}\right.$$

$$\left.\frac{w_1\mathcal{T}_1^{-(v_1/h)}}{\sigma_0^{\rho_1}}, \frac{w_2\mathcal{T}_1^{-(v_2/h)}}{\sigma_0^{\rho_2}}, \cdots, \frac{w_r\mathcal{T}_1^{-(v_r/h)}}{\sigma_0^{\rho_r}}, (c/2)\frac{\mathcal{T}_1^{-(\ell_1/h)}}{\sigma_0^{\ell_2}}\right]$$

where

(i) $\mathcal{T}_1 = \mathcal{T}/\sigma_0$

(ii) $\eta_t = (b_\ell + t)/\beta_\ell(\ell = 1, \cdots, m)$

(iii) R stands for

$$[1-(1/h)\{\sum_1^r(\alpha_k/\sigma_k + \eta_t(u_k/\sigma_k)) + \rho\}, v_1, v_2, \cdots, v_r, \ell_1],$$

$$[1-\{(1-h)\sum_1^r(\alpha_k/\sigma_k + \eta_t(u_k/\sigma_k)) + \rho/h\}, (\rho_1 - v_1/h), (\rho_2 - v_2/h),$$

$$\cdots, (\rho_r - v_r/h), (\ell_2 - \ell_1/h)], \{[a]; \theta_1', \theta_2'', \cdots, \theta_r^{(r)}, 1\}$$

(iv) S stands for

$$\{[c]; \psi', \psi'', \cdots \psi^{(r)}, 1\}, [1 - \{\sum_1^r(\alpha_k/\sigma_k - \eta_t(u_k/\sigma_k))\}; \rho_1, \cdots, \rho_r, \ell_2]$$

Also, above integral is valid under the following set of conditions:

(a) $c, \alpha_k, \sigma_k, \mathcal{T}, u_k, v_k, \rho_k (k = 1, \cdots, r), \sigma_0, h$ are all positive quantities having $v_k < \rho_k (k = 1, \cdots, r), \alpha > -1$, and the H–function of several variables in (2) satisfy conditions corresponding appropriately to those given by Srivastava and Panda [17].

(b) $Re[\rho + \sum_{k=1}^r(\alpha_k/\sigma_k + (b_w/\beta_w)(u_k/\sigma_k) + \sum_{k=1}^r v_k(d_j^{(k)}/\delta_j^{(k)})] > 0$

$$Re[\rho + \sum_{k=1}^r(v_k - \rho_k)(d_j^{(k)}/\delta_j^{(k)})] < 0, (w = 1, \cdots, m, j = 1, \cdots, A)$$

$$D = \sum_{j=1}^n \epsilon_j - \sum_{n+1}^p \epsilon_j + \sum_1^m \beta_j - \sum_1^q \beta_j > 0$$

$$|arg\, b| < (1/2)\cdot D\cdot\pi, \sum_1^q \beta_j - \sum_1^p \epsilon_j > 0$$

(c) The series occurring on the right–hand side of (19) is absolutely convergent.

Proof. We take

$$f(z) = z^{\rho}\psi_{\alpha_n}[c, z^{\ell_1}/(\sigma_0 + \mathcal{T} z^h)^{\ell_2}]H\left[\frac{w_1 z^{v_1}}{(\sigma_0 + \mathcal{T} z^h)^{\rho_1}}, \cdots, \frac{w_r z^{v_r}}{(\sigma_0 + \mathcal{T} z^h)^{\rho_r}}\right] \tag{20}$$

Now substituting this value of $f(z)$ in (16), expressing H–function of r–variables with the help of (2), changing the order of z–integral and multiple contour integrals (which is permissible under the conditions stated), we obtain :

$$\begin{aligned} I &= \prod_1^r(\sigma_k^{-1})\frac{1}{(2\pi i)^r}\int_{L_1}\int_{L_2}\cdots\int_{L_r}\psi(s_1, s_2, \cdots, s_r)\prod_{k=1}^{r}[\varphi_k(s_k)w_k^{s_k}] \\ &\times\left\{\int_0^{\infty}\frac{z^{\sum_1^r(\alpha_k/\sigma_k+v_k s_k)+\rho-1}}{(\sigma_0 + \mathcal{T} z^h)^{\sum_1^r(\alpha_k/\sigma_k+\rho_k s_k)}}H_{p+r,q+1}^{m,n+r}\left[\frac{bz^{\sum_1^r(u_k/\sigma_k)}}{(\sigma_0 + \mathcal{T} z^h)^{\sum_1^r(u_k/\sigma_k)}}\right.\right. \\ &\left.\left.\left|\begin{matrix}\cdots\cdots\\ \cdots\cdots\end{matrix}\right.\right]\times\psi_{\alpha_n}(c, z^{\ell_1}/(\sigma_0 + \mathcal{T} z^h)^{\ell_2})\right\}ds_1 ds_2\cdots ds_r \end{aligned} \tag{21}$$

where $\cdots$ indicate that the parameters of H–function in the above equation are the same as in the result (16) and $\psi(s_1, s_2, \cdots, s_r), \varphi_k(s_k), (k = 1, \cdots, r)$ stand for the quantities mentioned in (3) and (4), respectively.

Now in order to evaluate the z–integral in (21), we express the spheroidal function

$$\psi_{\alpha_n}[c, z^{\ell_1}/(\sigma_0 + \mathcal{T} z^h)^{\ell_2}]$$

as (9) and the H–function involved therein, in terms of Mellin–Barne's integral by definition (1), interchange the order of z and (ξ_1, ξ_2)–integrals, also interchange the summation and integrations, evaluate the z–integral thus obtained by using the well known formula (15) and interpret the result so obtained again by virtue of (2); we get under the conditions stated with (19),

$$\begin{aligned} I &= \frac{i^{\nu}\sqrt{\pi}2^{-(\alpha+1)}}{\prod_1^r(\sigma_k)v_{\alpha\nu}(c)}\sum{}^{*}{}_{k=0,1}^{\infty}a_k(c|\alpha\nu)\left\{\frac{1}{(2\pi i)^{r+1}}\int_{L_1}\int_{L_2}\cdots\int_{L_r}\int_{L_{r+1}}\psi(s_1, \cdots, s_r)\right. \\ &\prod_1^r[\psi_k(s_k)w_k^{s_k}](c/2)^{\xi_1}\frac{\Gamma((1/2)k - (1/2)^{\xi_1})}{\Gamma(\alpha + k/2 + 3/2 + (1/2)^{\xi_1})} \\ &\Gamma[\sum_1^r(\rho_k - v_k)s_k + (\ell_1 - \ell_2)\xi_1 - \rho]\cdot\mathcal{T}_1^{(-1/h)[\sum_1^r(\alpha_k/\sigma_k+v_k s_k)+\ell_1\xi_1+\rho]} \\ &\left.\cdot\sigma_0^{-[\sum_1^r(\alpha_k/\sigma_k+\rho_k s_k)+\ell_2\xi_2]}\times H_{p+r+2,q+2}^{m,n+r+2}\left[\frac{b\sigma_0^{-\sum_1^r(u_k/\sigma_k)}}{\mathcal{T}_1^{(1/h)\sum_1^r(u_k/\sigma_k)}}\left|\begin{matrix}L\\ M\end{matrix}\right.\right]ds_1 ds_2\cdots ds_r d\xi_1\right\} \end{aligned} \tag{22}$$

where for convenience,

(i) $\mathcal{T}_1 = \mathcal{T}/\sigma_0$.

(ii) L stands for

$$[1-(1/h)\{\sum_1^r(\alpha_k/\sigma_k+v_ks_k)+\ell_1\xi_2+\rho\},\sum_1^r u_k/\sigma_k],(1-\alpha_1/\sigma_1,u_1/\sigma_1),$$
$$\cdots,(1-\alpha_r/\sigma_r,u_r/\sigma_r),(a_j,\epsilon_j)_{1,p},[1-\{(1-1/h)\sum_1^r(\alpha_k/\sigma_k)+$$
$$(\ell_2-\ell_1/h)\xi_2+\sum_1^r(\rho_k-v_k/h)s_k-\rho/h\},(1-1/h)\sum_1^r(u_k/\sigma_k)]$$

(iii) M stands for

$$(b_j,\beta_j)_{1,q},[1-\sum_1^r(\alpha_k/\sigma_k),\sum_1^r(u_k/\sigma_k)],[1-\{\sum_1^r(\alpha_k/\sigma_k+\rho_ks_k)$$
$$+\ell_2\xi_2\},\sum_1^r(u_k/\sigma_k)]$$

Now expressing the function $H^{m,n+r+2}_{p+r+2,q+2}[\cdots]$ in terms of a series of gamma functions with the help of the well–known formula (13), interchanging the orders of integration and summation (which is permissible) and interpreting the resulting multple integral with the help of (2), we arrive at the desired result (19).

The change of order of integration and summation is justified [2] due to absolute convergence of the integrals and uniform convergence of the series involved in each process.

4. Convergence of Series Involved in (19)

Regarding the convergence of the series on the right–hand sides of the results (19), it is important to note that the ratio $a_{k+2}(c|\alpha_n)/a_k(c|\alpha_n)$ is $-c^2/4k^2$ for large k [14] and the ratio of the gammas involving k and t is bounded for large values of k (even or odd) and t by virtue of the fairly well–known result (cf. [11], p. 47):

$$\frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)} = k^{\alpha-\beta}[1+\mathbf{O}(k^{-1})], \quad k\to\infty.$$

Hence the series on the right–hand side of (19) are proved to be uniformly and absolutely convergent by M–test [2].

5. Discussion

The spheroidal wave functions $S_{mn}(c,x)$ and the periodic Mathieu functions $ce_n(\cos^{-1}x/2,c^2), se_{n+1}(\cos^{-1}x/2,c^2)$ are special cases related to $\psi_{\alpha_n}(c,x)$ as follows: ([14], eqs. (18), (23), and (25)):

$$\psi_{\alpha_n}(c,x)=\begin{cases}(1-x^2)^{-m/2}S_{mn}(c,x), & \alpha=m=0,1,\cdots\\ ce_n(\cos^{-1}x/2,c^2), & \alpha=-1/2\\ (1-x^2)^{-1/2}se_{n+1}(\cos^{-1}x/2,c^2), & \alpha=1/2.\end{cases}$$

Thus, by virtue of above properties of $\psi_{\alpha_n}(c,x)$, new results corresponding to our result (16) and (19) can be easily deduced. However, we do not mention them here for lack of space.

Since the H–functions occurring in the integrand of (16) and (19) includes a large variety of special functions as pointed out by Gupta and Jain [9] and Gupta, Kalla and Handa [10], a number of interesting integrals can be obtained as special cases of our integrals merely by specializing the parameters of the functions involved therein.

If we take $\sigma_0=1$, and $h=1$ in (16) and (19) the results recently obtained by Sharma and Salar [15] follow as particular cases.

Moreover, the convergence of double infinite series occurring on the right–hand side of our results (19) has been particularly examined.

Since the spheroidal functions are being used in such diverse fields as anteena theory [3], stochastic process [16], and the statistical theory of energy levels of complex systems [4] etc. Thus, we may find important applications of these results in various fields of mathematical physics, statistics, and engineering etc.

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