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ON SEMIPRIME SEMIGROUP RINGS

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When $R[S]$ is a commutative semigroup ring, some interesting conditions of $R[S]$ to be semiprime or von Neumann regular have been studied in [4].

For noncommutative semigroup rings, we partially generalize these results for semigroup rings with a polynomial identity. Indeed, when the coefficient ring R is P.I., semiprime or von Neumann regular semigroup ring $R[S]$ will be characterized.

A ring is semiprime if there is no nonzero nilpotent ideal. We say that a ring R is a P.I.–ring if R satisfies a polynomial identity with coefficients in the center and at least one coefficient is invertible.

Due to I. Kaplansky, a (left) primitive P.I.ring is simple which is finite dimensional over its center. When R is semiprime P.I., by L. Rowen [7] every nonzero ideal of R intersects the center $Z(R)$ of R nontrivially.

A ring R is called von Neumann regular if for every a in R there is b in R such that $a = aba$. Equivalently R is von Neumann regular if and only if every finitely generated one–sided ideal is generated by an idempotent. For more detail, see [5].

For a commutative monoid S, elements a and b of S are *asymptotically equivalent* if there exists a positive integer n_0 such that $na = nb$ for each $n \geq n_0$. The monoid S is said to be *free of asymptotic torsion* if any two distinct elements of S are not asymptotically equivalent.

We start with a characterization of semiprime semi–group ring.

Theorem 1. *Let R be a P.I.-ring and S be a commutative monoid. Then the semigroup ring $R[S]$ is a semiprime ring if and only if the following*

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three conditions are satisfied:

(1) *R is semiprime.*

(2) *S is free of asymptotic torsion.*

(3) *p is regular in R for each prime p such that S is not p-torsion free.*

Proof. Assume that $R[S]$ is semiprime. Then obviously the center $Z(R[S]) = Z(R)[S]$ of $R[S]$ is semiprime. So by R. Gilmer's result, condition (2) is satisfied. Also p is regular in $Z(R)$ for each prime p such that S is not p–torsion free. Furthermore, it can be easily checked that R is semiprime. Now for condition (3), let p be a prime number such that S is not p–torsion free. Then $J = \{x \in R \,|\, px = 0\}$ is an ideal of R. If $J \neq 0$, then $J \cap Z(R) \neq 0$ by L.Rowen [7]. Thus there is a non–zero element y in $J \cap Z(R)$, that is, $py = 0$ with $0 \neq y$ in $Z(R)$. But this is a contradiction because p is regular in $Z(R)$. Therefore $J = 0$ and hence p is regular in R, so we have our condition (3).

Conversely, assume that conditions (1), (2) and (3) are satisfied. To show that $R[S]$ is semiprime, let I be an ideal of $R[S]$ with $I^2 = 0$. Now if $I \neq 0$, then we can choose a nonzero element α in I with the minimal length n; say

$$\alpha = a_1 s_1 + a_2 s_2 + \cdots + a_n s_n$$

with $0 \neq a_i \in R$, $s_i \in S$, $i = 1, 2, \cdots, n$. So Ra_nR is a nonzero ideal of R. But since R is semiprime P.I., $Ra_nR \cap Z(R) \neq 0$. Therefore there are $r_1, r_2, \cdots, r_k$ and $r'_1, r'_2, \cdots r'_k$ in R such that

$$0 \neq r_1 a_n r'_1 + r_2 a_n r'_2 + \cdots + r_k a_n r'_k$$

is in $Z(R)$. Thus

$$\begin{aligned} \beta &= r_1 \alpha r'_1 + r_2 \alpha r'_2 + \cdots + r_k \alpha r'_k \\ &= (\sum_{i=1}^{k} r_i a_1 r'_i) s_1 + \cdots + (\sum_{i=1}^{k} r_i a_n r'_i) s_n \end{aligned}$$

is in I. Setting $b_1 = \sum_{i=1}^{k} r_i a_1 r'_i, \cdots, b_n = \sum_{i=1}^{k} r_i a_n r'_i$. Then $\beta = b_1 s_1 + b_2 s_2 + \cdots + b_n s_n$ with $0 \neq b_n \in Z(R)$.
Now for any r in R,

$$r\beta - \beta r = (rb_1 - b_1 r)s_1 + \cdots + (rb_n - b_n r)s_n$$

is in I. In this case, since $b_n \in Z(R), rb_n - b_n r = 0$. But since the minimal length of nonzero elements in I is n, we have $rb_1 - b_1 r = 0, \cdots, rb_{n-1} -$

$b_{n-1}r = 0$. So $r\beta - \beta r = 0$ for any r in R and therefore $\beta \in I \cap Z(R)[S]$ with $\beta \neq 0$. By this fact $J = I \cap Z(R)[S]$ is a nonzero ideal of $Z(R)[S]$ satisfying $J^2 = (I \cap Z(R)[S])^2 \subseteq I^2 = 0$.

On the other hand, by our given conditions (1), (2) and (3), $Z(R)$ is semiprime because R is semiprime and S is free of asymptotic torsion. Also from condition (3), p is regular in $Z(R)$ for each prime p such that S is not p–torsion free. By Gilmer's result, $Z(R)[S]$ is semiprime. But since J is a nilpotent ideal, we have $J = 0$, which is a contradiction. Therefore $R[S]$ has no nonzero nilpotent ideal, that is, $R[S]$ is semiprime.

For von Neumann regular semigroup rings, we have following

Proposition 2. *Let R be a P.I.-ring and S be a commutative monoid. If the semigroup ring $R[S]$ is von Neumann regular, then the following conditions are satisfied:*

(1) *R is a von Neumann regular ring.*
(2) *S is free of asymptotic torsion.*
(3) *p is regular in R for each prime such that S is not p–torsion free.*
(4) *S is periodic.*

Proof. Obviously that the ring R is an isomorphic image of the ring $R[S]$ via the augmentation map, R is von Neumann regular. For other conditions, let M be a maximal ideal of R. Then by the naturally induced map from $R[S]$, the ring $(R/M)[S]$ is also a von Neumann regular ring. By I. Kaplansky, the simple P.I. ring R/M is isomorphic to $n \times n$ matrix ring $\mathrm{Mat}_n(D)$ for some positive integer n and for some division ring D which is finite dimensional over its center F.

In this case $(R/M)[S]$ $=\mathrm{Mat}_n(D[S])$ and so $D[S]$ is von Neumann regular, let $\{u_1, u_2, \cdots, u_k\}$ be a basis of D over F. Then $\{u_1, u_2, \cdots, u_k\}$ is also a finite centralizing basis of $D[S]$ over $F[S]$. So the von Neumann regularity of $D[S]$ implies that of $F[S]$. Now by Gilmer's result, we get conditions (2) and (4).

Finally for condition (3), let p be a prime number such that S is not p–torsion free. For a given maximal ideal M of R, let $I = \{x \in R/M \mid px = 0\}$. Then I is an ideal of the simple ring R/M. Let F be the center of R/M. Then as we already observed, the ring $F[S]$ is von Neumann regular. Now if $I \neq 0$, then $I = R/M$ and so p is a zero divisor in F, which is a contradiction by Gilmer. Thus $I = 0$. That is, p is regular in R/M for any maximal ideal M of R. Since R is von Neumann regular P.I., the intersection of all maximal ideals of R is zero. So the ring R has a representation as a subdirect sum of simple rings R/M with M a maximal ideal. Hence p is regular in R such that S is not p–torsion free and so we have condition (3).

Proposition 3. *Let R be an Azumaya algebra and S be a commutative monoid. If R and S satisfy four conditions of Proposition 2, then $R[S]$ is a von Neumann regular ring.*

Proof. First we claim that the ring $R[S_0]$ is a von Neumann regular ring for a finite submonoid S_0 of S. By our hypothesis, since R is von Neumann regular, the center $Z(R)$ of R is also a von Neumann regular ring. Note that the finite submonoid S_0 of S also satisfies conditions (2), (3) and (4) because S already satisfied these conditions. Thus by Gilmer's result, the ring $Z(R)[S_0]$ is a von Neumann regular ring. Now by assumption, since R is Azumaya, R is a finitely generated as $Z(R)$–module, and so $R[S_0]$ is finitely generated as a $Z(R)[S_0]$–module. By Theorem 1, $R[S_0]$ is semiprime ring. Hence by Armendariz, $R[S_0]$ is a von Neumann regular ring.

Finally to show that $R[S]$ is von Neumann regular, let $\alpha = a_1s_1+a_2s_2+\cdots+a_ns_n$ be an element of $R[S]$ with $a_i \in R$ and $s_i \in S$, $i = 1, 2, \cdots, n$. Thus α is in $R[S_0]$, where S_0 denotes the submonoid of S generated by $s_1, s_2, \cdots, s_n$. But since S is commutative and periodic by the condition (4), the submonoid S_0 is finite. Since $R[S_0]$ is von Neumann regular, there exists β in $R[S_0]$ such that $\alpha = \alpha\beta\alpha$. Thus $R[S]$ is von Neumann regular.

Combining Proposition 2 and 3, we are able to generalize Gilmer's result to a class of some interesting noncommutative rings.

Theorem 4. *Let R be an Azumaya algebra and S be a commutative monoid. Then the semigroup ring $R[S]$ is a von Neumann regular if and only if the following conditions are satisfied:*

(1) *R is a von Neumann regular ring.*

(2) *S is free of asymptotic torsion.*

(3) *p is regular in R for each prime number p such that S is not p-torsion free.*

(4) *S is periodic.*

References

[1] E.P. Armendariz, *On semiprime P.I.-algebra over commutative regular rings*, Pacific J. Math., 66 (1976), 23-28.

[2] E.P. Armendariz and J.W. Fisher, *Regular P.I.-rings*, Proc. Amer. Math. Soc., 39 (1973), 247-251.

[3] J.W. Fisher and R.L. Snider, *On the von Neumann regularity of rings with regular prime factor rings*, Pacific J. Math., 54(1974), 135-144.

[4] R. Gilmer, *Commutative Semigroup Rings*, The University of Chicago Press, Chicago, 1984.

[5] K.R. Goodearl, von Neumann Regular Rings, Pitnam, London and San Francisco, 1979.

[6] I.N. Herstein, *Non-commutative Rings*, Carus Monograph, 15, 1973.

[7] L. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York, 1983.

[8] ________, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc., 79 (1973), 219-223.

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