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ON DILATION THEOREMS OF A CONTRACTION IN THE CLASSES $\mathbf{A}_n$

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Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $I_{\mathcal{H}}$ and is closed in the ultraweak operator topololgy. Moreover, let $\mathcal{Q}_T$ denote the quotient space $\mathcal{C}_1/{}^{\perp}\mathcal{A}_T$, where $\mathcal{C}_1$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and ${}^{\perp}\mathcal{A}_T$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1$. One knows that $\mathcal{A}_T$ is the dual space of $\mathcal{Q}_T$ and that the duality is given by

$$(1) \qquad \langle A, [L]\rangle = tr(AL), \quad A \in \mathcal{A}_T, [L] \in \mathcal{Q}_T.$$

Furthermore, the weak* topology that accrues to $\mathcal{A}_T$ by virtue of this duality coincides with the ultraweak operator topology on $\mathcal{A}_T$. For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $\mathcal{C}_1$ defined by

$$(2) \qquad (x \otimes y)(u) = (u, y)x, \quad u \in \mathcal{H}.$$

The theory of dual algebras is deeply related to the study of the problem of solving systems of simultaneous equations in the predual of a dual algebra (cf. [1], [3], [5], and [7]). That is the main topic of this work. In this paper, we consider the following question:

Question 1. *Let A be a normal completely nonunitary contraction acting on an n-dimensional Hilbert space such that $\|Ax\| < \|x\|$ for every nonzero vector x and let $T \in \mathbf{A}_m(\mathcal{H})$ (will be defined below), where $m = n(n+1)/2$.*

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Is it true that there always exist invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T with $\mathcal{M} \supset \mathcal{N}$ such that the compression $T_{\mathcal{M}\ominus\mathcal{N}}$ of T to $\mathcal{M} \ominus \mathcal{N}$ is unitarily equivalent to A?

The notation and terminology employed herein agree with those in [5],[6], and [8]. We shall denote by $\mathbf{D}$ the open unit disc in the complex plane $\mathbf{C}$, and we write $\mathbf{T}$ for the boundary of $\mathbf{D}$. For $1 \le p < \infty$, we denote by $L^p = L^p(\mathbf{T})$ the Banach space of complex valued, Lebesgue measurable functions f on $\mathbf{T}$ such that $|f|^p$ is Lebesgue integrable, and by $L^\infty = L^\infty(\mathbf{T})$ the Banach algebra of all complex valued Lebesgue measurable, essentially bounded functions on $\mathbf{T}$. If for $1 \le p \le \infty$ we denote by $H^p = H^p(\mathbf{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish, then one knows that the preannihilator ${}^\perp(H^\infty)$ of H^∞ in L^1 is the subspace H_0^1 consisting of those functions g in H^1 whose analytic extension $\tilde{g}$ to $\mathbf{D}$ satisfies $\tilde{g}(0) = 0$. It is well known that H^∞ is the dual space of L^1/H_0^1, where the duality is given by the pairing

$$(3) \qquad \langle f, [g] \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) g(e^{it}) dt, \quad f \in H^\infty, \quad [g] \in L^1/H_0^1.$$

Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0), T will be called an *absolutely continuous contraction.* The following Foias–Sz.–Nagy functional calculus [5, Theorem 4.1] provides a good relationship between the function space H^∞ and a dual algebra $\mathcal{A}_T$.

Theorem 2. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ that has the following properties:*

(a) $\Phi_T(1) = 1_{\mathcal{H}}, \Phi_T(\xi) = T$,

(b) $\|\Phi_T(f)\| \le \|f\|_\infty, f \in H^\infty$,

(c) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*

(d) *the range of Φ_T is weak* dense in $\mathcal{A}_T$,*

(e) *there exists a bounded, linear, one-to-one map $\phi_T : \mathcal{Q}_T \to L^1/H_0^1$ such that $\phi_T^* = \Phi_T$, and*

(f) *if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of $\mathcal{Q}_T$ onto L^1/H_0^1.*

Definition 3 (cf. [4]). Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let n be any cardinal number such that $1 \le n \le \aleph_0$. Then $\mathcal{A}$ will be said to have property $(\mathbf{A}_n)$ provided every $n \times n$ system of simultaneous equations of the form

$$[L_{ij}] = [x_i \otimes y_j], \quad 0 \le i, j < n \tag{4}$$

(which the $[L_{ij}]$ are arbitrary but fixed elements from $\mathcal{Q}_{\mathcal{A}}$) has a solution $\{x_i\}_{0 \le i < n}$, $\{y_j\}_{0 \le j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$.

Definition 4 (cf. [4]). The class $\mathbf{A}(\mathcal{H})$ consists of all those absolutely continuous contraction T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, if n is any cardinal number such that $1 \le n \le \aleph_0$, we denote by $\mathbf{A}_n(\mathcal{H})$ the set of all T in $\mathbf{A}(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property $(\mathbf{A}_n)$.

We write simply $\mathbf{A}_n$ for $\mathbf{A}_n(\mathcal{H})$ when there is no confusion. If $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{M} \subset \mathcal{H}$ is a semi-invariant subspace for T (i.e., there exist invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ for T with $\mathcal{N}_1 \supset \mathcal{N}_2$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2 = \mathcal{N}_1 \cap \mathcal{N}_2^\perp$), we write $T_{\mathcal{M}}$ for the compression of T to $\mathcal{M}$. In other words, $T_{\mathcal{M}} = P_{\mathcal{M}} T | \mathcal{M}$, where $P_{\mathcal{M}}$ is the orthogonal projection whose range is $\mathcal{M}$. Let n be any cardinal number such that $1 \le n \le \aleph_0$. Throughout this paper, we write $\mathbf{C}$ for the complex plane and $\mathbf{N}$ for the set of natural numbers. Now we are ready to show the main theorem of this paper.

Theorem 5. *Let A be a completely nonunitary normal contraction acting on an n-dimensional Hilbert space $\mathcal{H}_n$, $2 \le n \in \mathbf{N}$, whose matrix relative to some orthonormal basis $\{u_k\}_{k=1}^n$ for $\mathcal{H}_n$ is the diagonal matrix* $\mathrm{Diag}(\{\lambda_k\})_{k=1}^n$ *and let $T \in \mathbf{A}_m(\mathcal{H})$, where $m = n(n+1)/2$. Then there exist invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T with $\mathcal{M} \supset \mathcal{N}$ such that the compression $T_{\mathcal{M} \ominus \mathcal{N}}$ of T to $\mathcal{M} \ominus \mathcal{N}$ is unitarily equivalent to A.*

Proof. Let $\mathcal{H}_m$ be an m-dimensional Hilbert space. We define a normal operator $\widetilde{N} \in \mathcal{L}(\mathcal{H}_m)$ whose matrix relative to some orthonormal basis $\{u_k^{(i)}\}_{\substack{1 \le i \le k \\ 1 \le k \le n}}$ for $\mathcal{H}_m$ is a diagonal matrix

$$\mathrm{Diag}(\lambda_1^{(1)}, \underbrace{\lambda_2^{(1)}, \lambda_2^{(2)}}_{(2)}, \underbrace{\lambda_3^{(1)}, \lambda_3^{(2)}, \lambda_3^{(3)}}_{(3)}, \cdots, \underbrace{\lambda_n^{(1)}, \cdots, \lambda_n^{(n)}}_{(n)}), \tag{5}$$

where $\lambda_k^{(1)} = \lambda_k^{(2)} = \cdots = \lambda_k^{(k)} = \lambda_k$, for $k = 1, 2, \cdots, n$. Since $\widetilde{N}$ is a completely nonunitary contraction, we have $\{\lambda_k\}_{k=1}^n \subset \mathbf{D}$ and it follows from [4, Corollary 3.5] that there exist invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T with $\mathcal{M} \supset \mathcal{N}$ such that $\dim(\mathcal{M} \ominus \mathcal{N}) = m$ and $T_{\mathcal{M} \ominus \mathcal{N}}$ is similar to $\widetilde{N}$.

Let X be an invertible operator with $T_{\mathcal{M}\ominus\mathcal{N}}X = X\widetilde{N}$. Note that

$$\widetilde{N}u_k^{(i)} = \lambda_k^{(i)}u_k^{(i)}, \quad 1 \le i \le k, \quad 1 \le k \le n. \tag{6}$$

For a brief notation, we write $\widetilde{T} = T_{\mathcal{M}\ominus\mathcal{N}}$. Since X is one–to–one, it is easy to show that there exists a linearly independent set $\{w_k^{(i)}\}_{\substack{1\le i\le k\\ 1\le k\le n}}$ in $\mathcal{M}\ominus\mathcal{N}$ such that $\|w_k^{(i)}\| = 1$ and

$$\widetilde{T}w_k^{(i)} = \lambda_k^{(i)}w_k^{(i)}, \quad 1 \le i \le k, \quad 1 \le k \le n. \tag{7}$$

Taking $f_1 = w_1^{(1)}$, we have $\widetilde{T}f_1 = \lambda_1 f_1$. Assume that there exist $f_1, \cdots, f_k$ in $\mathcal{M}\ominus\mathcal{N}$ with $k < n$ such that $\widetilde{T}f_i = \lambda_i f_i, i = 1, \cdots, k$. Since $\{w_{k+1}^{(1)}, \cdots, w_{k+1}^{(k+1)}\}$ induces an $(k+1)$–dimensional Hilbert space $\mathcal{R}$, there exists a normal vector $f_{k+1} \in \mathcal{R}$ such that $(f_i, f_{k+1}) = 0, \quad i = 1, 2, \cdots, k$. Say

$$f_{k+1} = \sum_{i=1}^{k+1} a_i w_{k+1}^{(i)}, \tag{8}$$

where $a_i \in \mathbf{C}$, $i = 1, \cdots, k+1$. Then we have

$$\begin{aligned} \widetilde{T}f_{k+1} &= \widetilde{T}(\sum_{i=1}^{k+1} a_i w_{k+1}^{(i)}) \\ &= \sum_{i=1}^{k+1} a_i \widetilde{T}w_{k+1}^{(i)} \\ &= \sum_{i=1}^{k+1} a_i \lambda_{k+1}^{(i)} w_{k+1}^{(i)} \\ &= \lambda_{k+1}\sum_{i=1}^{k+1} a_i w_{k+1}^{(i)} \\ &= \lambda_{k+1} f_{k+1}. \end{aligned} \tag{9}$$

Hence by the mathematical induction, there exists a set $\{f_i\}_{i=1}^n \subset \mathcal{M}\ominus\mathcal{N}$ such that $\widetilde{T}f_i = \lambda_i f_i$, for $i = 1, 2, \cdots, n$. Let us denote

$$\mathcal{K} = \bigvee_{k=1}^{n} f_k. \tag{10}$$

If we define a linear map $Y : \mathcal{H}_n \to \mathcal{K}$ with $Yu_k = f_k, \quad k = 1, 2, \cdots, n$, then it is obvious that Y is onto and isometry. Since $\mathcal{K}$ is an invariant subspace for $\tilde{T}$, $\mathcal{K}$ is a semi–invariant subspace for T. Furthermore, we have $T_{\mathcal{K}}Y = YA$. Hence A is unitarily equivalent to $T_{\mathcal{K}}$ and the proof is complete.

Remark 6. Theorem 5 gives a solution for Question 1.

Remark 7. It follows from [4, Corollary 3.6] that if $\lambda \in \mathbf{D}$ and $T \in \mathbf{A}_n$, then there exist invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T with $\mathcal{M} \supset \mathcal{N}$ such that $\dim(\mathcal{M} \ominus \mathcal{N}) = n$ and $T_{\mathcal{M}\ominus\mathcal{N}} = \lambda I$. This statement is a special case for the work of this paper.

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