Kyungpook Mathematical Journal
Volume 30, Number 1, June, 1990

TOPOLOGIES ON GENERALIZED SEMI–INNER PRODUCT ALGEBRAS OF TYPE (p)

Mohammed A. Abohadi

1. Introduction

Using the concept of Prugovecki [10] and Lumer [6], Nath [9] introduce what he called generalized semi-inner product spaces, and studied strong topologies on these spaces. On the other hand Ambrose [1] introduced and studied a special class of Banach algebras called H^*–algebras whose underlying spaces are Hilbert spaces. Later on, Husain and Malyviya [4] replaced the Hilbert space structure in H^*–algebras by a more general structure called semi–inner product space and they obtained a new class of algebras called semi–inner product algebras. This concept led Husain and Khaleelulla [5] to define the concept of a generalized semi–inner product algebra and they obtained certain results on such algebras.

Using this concept and the concept of Elsayyad [3], we introduce, in the present paper, the concept of a generalized semi–inner product algebra of type (p) and study strong and weak topologies on such algebras which make them locally convex as well as locally m–convex.

2. Preliminaries

Definition 2.1 [8]. Let E be a vector space. We define a map $[.,.]$: $E \times E \to k$ satisfying the following conditions:

(S_1) $[x+y,z] = [x,z] + [y,z]$, x, y and $z \in E$.

(S_2) $[\lambda x, y] = \lambda[x,y]$, $\lambda \in k$.

(S_3) $[x,x] > 0$, if $x \neq 0$.

(S_4) $|[x,y]| < [x,x]^{\frac{1}{p}}[y,y]^{\frac{p-1}{p}}, 1 < p < \infty$.

Received January 13, 1987

Then $[.,.]$ is called a *semi-inner product* of type (p) on E, and E, equipped with the map $[.,.]$, is called a semi–inner product space of type (p) (abbriviated as s.i.p. space of type (p).

Remark 2.2. If $p = 2$, this concept is called semi–inner product space which is due to Lumer [6] (abbriviated as s.i.p.space).

Remark 2.3. Nath [8] proved that a s.i.p space of type (p) becomes a normed space under $\|x\| = [x,x]^{\frac{1}{p}}$ and a normed space can be made into s.i.p. space of type (p).

Definition 2.4 [2]. A normed algebra E is a normed space with is also algebra such that $\|xy\| \leq \|x\|\|y\|$ for all x and $y \in E$.

Definition 2.5 [7]. Let E be an algebra.

(a) A subset V of E is called an idempotent if $VV \subset V$.

(b) A subset V of E is called m–convex (multiplicatively convex) if V is convex and idempotent.

Definition 2.6 [7]. A locally convex algebra is an algebra and a Hausdorff locally convex space.

Definition 2.7 [7]. A locally convex algebra is called locally m–convex algebra if there exists a neighbourhood basis of consisting of m–convex sets.

Definition 2.8 [5]. A vector space E is called semi–inner product algebra (abbriviated as s.i.p. algebra) if

(i) E is a normed algebra,

(ii) E is s.i.p. space with the same norm as that of normed algebra.

Definition 2.9 [3]. A vector space E is called semi–inner product algebra of type (p) (abbriviated as s.i.p. algebra of type (p)) if

(i) E is a normed algebra,

(ii) E is s.i.p. space of type (p) with the same norm as that of normed algebra.

Remark 2.10. In [3], E was assumed to be complete.

3. Generalized Semi–Inner Product Algebra of Type (p)

Definition 3.1 [5]. A vecotr space E is called a generalized semi–inner product algebra (abbriviated as g.s.i.p. algebra) if

(G_1) E is an algebra
(G_2) there is a subspace M of E which is s.i.p. algebra,
(G_3) there is a set L of linear multiplicative operators of E satisfying:
(i) each member of L maps E into M,
(ii) if $Tx = 0$ for all $T \in L$, then $x = 0$.

We denote a generalized semi–inner product algebra by the triple (E, L, M).

Remark 3.2. Every s.i.p. algebra is a g.s.i.p. algebra, with $M = E$ and $L = [I]$, I the identity operator on E.

Definition 3.3. A vector space E is called a generalized semi–inner product algebra of type (p) (abbriviated as g.s.i.p. algebra of type (p)) if
(G_1) E is an algebra,
(G_2) there is a subspace M of E which is s.i.p. algebra of type (p),
(G_3) there is a set of linear multiplicative operators on E satisfying:
(i) each member of L maps E into M,
(ii) if $Tx = 0$ for all $T \in L$, then $x = 0$.

We denote a generalized semi–inner product algebra of type (p) by the triple (E, L, M).

Remark 3.4. Every s.i.p. algebra of type (p) is a g.s.i.p. algebra of type (p), with $M = E$ and $L = [I]$, I the identity operator on E.

Remark 3.5. The example which was given in [5] and [9] is incorrect because the operator T^{p-1} is not linear.

Remark 3.6. It would be interesting to find a non–trivial example of a g.s.i.p. algebra of type (p) which is not s.i.p. algebra of type (p).

4. Strong Topology

Definition 4.1. Let (E, L, M) be a g.s.i.p. algebra of type (p). For each $x \in E$, the family of sets

$$V(x; T_1, \cdots, T_n; \varepsilon) = \{y \in E; [T_k(y - x), T_k(y - x)]^{\frac{1}{p}} < \varepsilon,$$

$k = 1, 2, \cdots, n\} \forall \varepsilon > 0$, $T_1, \cdots, T_n \in L$ and $n = 1, 2, \cdots$. constitutes a neighbourhood basis at x for a topology on E which we call the strong topology.

Lemma 4.2. *Each $V(0; T_1, \cdots, T_n; \varepsilon)$ is circled and convex.*

Proof. Let $V = V(0; T_1, \cdots, T_n; \varepsilon)$. To show that V is circled: Let $\lambda \in C$ with $|\lambda| \le 1$ and $x \in V$.

$$\begin{aligned}[T_k(\lambda x), T_k(\lambda x)]^{\frac{1}{p}} &= \|T_k(\lambda x)\| \\ &= |\lambda| \|T_k(x)\| \\ &< |\lambda|\varepsilon \le \varepsilon, \quad k = 1, 2, \cdots, n.\end{aligned}$$

Thus $\|T_k(\lambda x)\| < \varepsilon, k = 1, 2, \cdots, n$. Hence $\lambda x \in V$. So V is circled. To show that V is convex: Let $\lambda \in \mathbf{C}$, $0 < \lambda < 1$ and $x, y \in V$.

$$\begin{aligned}\|T_k[\lambda x + (1-\lambda)y]\| &= \|T_k(\lambda x) + T_k[(1-\lambda)y]\| \\ &= \|\lambda T_k(x) + (1-\lambda)T_k(y)\| \\ &< \|T_k(x)\| + (1-\lambda)\|T_k(y)\| \\ &< \lambda\varepsilon + (1-\lambda)\varepsilon = \varepsilon.\end{aligned}$$

Thus $\|T_k[\lambda x + (1-\lambda)y]\| < \varepsilon, k = 1, 2, \cdots, n$. Hence $\lambda x + (1-\lambda)y \in V$. So V is convex.

Lemma 4.3. *Each $V(0; T_1, \cdots, T_n; \varepsilon), 0 < \varepsilon \le 1$, is m-convex.*

Proof. Let $V = V(0; T_1, \cdots, T_n; \varepsilon)$. Clearly V is convex by Lemma (4.2). To show V is an idempotent i.e. $VV \subset V$. Let x and $y \in V$, $0 < \varepsilon \le 1$.

$$\begin{aligned}[T_k(xy), T_k(xy)]^{\frac{1}{p}} &= [T_k(x)T_k(y), T_k(x)T_k(y)]^{\frac{1}{p}}, \\ &\qquad k = 1, 2, \cdots, n. \\ &\le [T_k(x), T_k(x)]^{\frac{1}{p}} [T_k(y), T_k(y)]^{\frac{1}{p}} \\ &< \varepsilon^2 \le \varepsilon.\end{aligned}$$

Thus $\|T_k(xy)\| < \varepsilon, k = 1, 2, \cdots, n$. Hence $xy \in V$ for all x and $y \in V$, $0 < \varepsilon \le 1$. So V is an idempotent. Thus V is m–convex.

Lemma 4.4 *Let (E, L, M) be a g.s.i.p. algebra of type (p). If a topology on it is introduced in which the sets $V(x; T; \varepsilon)$ are neighbourhoods of $x, \forall \varepsilon > 0$, $T \in L$, then the resulting topological space is Hausdorff.*

Proof. Here $[Tx, Tx] = \|Tx\|^p$. Suppose E is not a Hausdorff space. Then there exists at least two points $x_1, x_2 \in E$, $x_1 \ne x_2$ for which any two neighbourhoods have common points. Thus for any two neighbourhoods $V(x_1; T; \frac{1}{n})$ and $V(x_2; T; \frac{1}{n})$ there exists at least one $y_n \in E$ such that

$$y_n \in V(x_1; T; \frac{1}{n}) \cap V(x_2; T; \frac{1}{n}).$$

So,

$$\|T(y_n - x_1)\| < \frac{1}{n}, \quad \text{and} \quad \|T(y_n - x_2)\| < \frac{1}{n}.$$

Now,

$$\begin{aligned}
\|T(x_1 - x_2)\| &= \|T(x_1 - y_n + y_n - x_2)\| \\
&= \|T(x_1 - y_n) + T(y_n - x_2)\| \\
&\leq \|T(x_1 - y_n)\| + \|T(y_n - x_2)\| \\
&< \frac{1}{n} + \frac{1}{n} = \frac{2}{n}
\end{aligned}$$

Since the above is true for any positive integer n, it follows that $T(x_1 - x_2) = 0$ which is true for any $T \in L$. Thus $x_1 - x_2 = 0$; So $x_1 = x_2$; which is a contradiction.

Theorem 4.5. *Let (E, L, M) be a g.s.i.p. algebra of type (p). Then (E, L, M) equipped with the strong topology is a locally convex algebra.*

Proof. Nath [9] prove that (E, L, M) is a Hausdorff locally convex space. To complete the proof we show that for any $V(x_0x; T_1, \cdots, T_n; \varepsilon)$, there exists $V(x; T_1, \cdots, T_n; \frac{\varepsilon}{\lambda})$,

$$\lambda = \max_{1 \leq k \leq n} (\lambda_k), \quad \lambda_k = [T_k(x_0), T_k(x_0)]^{\frac{1}{p}},$$

$k = 1, 2, \cdots, n$; such that

$$x_0 V(x; T_1, \cdots, T_n; \frac{\varepsilon}{\lambda}) \subset V(x_0 x; T_1, \cdots, T_n; \varepsilon).$$

Let $y \in V(x; T_1, \cdots, T_n; \frac{\varepsilon}{\lambda})$; Then $[T_k(y-x), T_k(y-x)]^{\frac{1}{p}} < \frac{\varepsilon}{\lambda}, k = 1, 2, \cdots, n$. Now,

$$\begin{aligned}
[T_k(x_0 y - x_0 x), T_k(x_0 y - x_o x)]^{\frac{1}{p}} &= [T_k x_0(y - x), T_k x_0(y - x)]^{\frac{1}{p}} \\
&= [T_k(x_0)T_k(y - x), T_k(x_0)T_k(y - x)]^{\frac{1}{p}} \\
&\leq [T_k(x_0), T_k(x_0)]^{\frac{1}{p}}[T_k(y - x), T_k(y - x)]^{\frac{1}{p}} \\
&< \lambda \frac{\varepsilon}{\lambda} = \varepsilon.
\end{aligned}$$

Thus $x_0 y \in V(x_0 x; T_1, \cdots, T_n; \varepsilon)$, for all $y \in V(x; T_1, \cdots, T_n; \frac{\varepsilon}{\lambda})$, and this proves $x_0 V(x; T_1, \cdots, T_n; \frac{\varepsilon}{\lambda}) \subset V(x_0 x : T_1, \cdots, T_n; \varepsilon)$.

Similarly, we can show that

$$V(x;T_1,\cdots,T_n;\frac{\varepsilon}{\lambda})x_0 \subset V(xx_0;T_1,\cdots,T_n;\varepsilon).$$

This shows that (E,L,M) is a locally convex algebra under the strong topology.

Theorem 4.6. *Let (E,L,M) be a g.s.i.p algebra of type (p). Then (E,L,M) equipped with the strong topology is a locally m-convex algebra.*

Proof. Let

$$\lambda = \max_{1\leq k\leq n}(\lambda_k), \lambda_k = [T_k(x_0),T_k(x_0)]^{\frac{1}{p}},$$

$k = 1,2,\cdots,n$.
If $\lambda \geq 1$, then the result follows from Lemma (4.2) and lemma (4.3).
If $\lambda < 1$, then the result follows from lemma (4.3), because we can show that for any $V(x_0x : T_1,\cdots,T_n;\varepsilon)$, there exists $V(x;T_1,\cdots,T_n;\varepsilon)$ such that

$$x_0V(x;T_1,\cdots,T_n;\varepsilon) \subset V(x_0x;T_1,\cdots,T_n;\varepsilon).$$

Theorem 4.7. *Let (E,L,M) be a g.s.i.p. algebra of type (p). Then (E,L,M) with strong topology is metrizable if there is a countable subset β of L with the following property: For each $T\in L$, there exists an $S\in\mathcal{L}$ such that $[Tx,Tx]^{\frac{1}{p}} \leq [Sx,Sx]^{\frac{1}{p}}$, $x\in E$, where $\mathcal{L}$ is the linear manifold generated by β.*

Proof. It is sufficient to show that the family of sets

$$\{(0;S_1,\cdots,S_n;\frac{1}{n});S_1,\cdots,\varepsilon S_n, \quad k,n=1,2,\cdots\}$$

is a neighbourhood basis at 0 for the strong topology.
For every $T\in L$, we can find and $S\in\mathcal{L}$ for which

$$V(0;S;\varepsilon)\subset V(0;T;\varepsilon). \tag{1}$$

Since $\|Tx\|\leq\|Sx\|$.
Clearly, we have $S=\lambda_1S_1+\cdots+\lambda_kS_k$, where $S_1,\cdots,S_k\in\beta$ and so, $\forall x\in E$,

$$\begin{aligned}[Sx,Sx]^{\frac{1}{p}} &= \|Sx\| = \|(\lambda_1S_1+\cdots+\lambda_kS_k)(x)\| \\ &\leq |\lambda_1|\|S_1x\|+\cdots+|\lambda_k|\|S_kx\|.\end{aligned} \tag{2}$$

Thus, if we choose an integer n such that

$$\frac{1}{n} < \frac{\varepsilon}{k|\lambda_1|}, \cdots, \frac{1}{n} < \frac{\varepsilon}{k|\lambda_k|}.$$

Then $x \in V(0; S_r; \frac{1}{n})$ implies $\|S_r x\| \leq \frac{1}{n} < \frac{\varepsilon}{k|\lambda_r|}$, $r = 1, 2, \cdots, k$. So, (2) becomes $\|Sx\| < \frac{\varepsilon}{k} + \cdots + \frac{\varepsilon}{k}$ (k times) $< k\frac{\varepsilon}{k} = \varepsilon$.
So, $x \in V(0; S_1; \frac{1}{n}) \cap \cdots \cap V(0; S_k; \frac{1}{n})$ implies $x \in V(0; S; \varepsilon)$. i.e., $x \in V(0; S_1, \cdots, S_k; \frac{1}{n})$ implies $x \in V(0; S; \varepsilon)$. So,

$$V(0; S_1, \cdots, S_k; \frac{1}{n}) \subset V(0; S; \varepsilon).$$

Using (1)

$$\begin{aligned} V(0; T; \varepsilon) &\supset V(0; S; \varepsilon) \\ &\supset V(0; S_1; \frac{1}{n}) \cap \cdots \cap V(0; S_k; \frac{1}{n}) \\ &= V(0; S_1, \cdots, S_k; \frac{1}{n}) \end{aligned}$$

Thus the family $\{V(0; S_1, \cdots, S_k; \frac{1}{n}): S_1, \cdots, S_k \in \beta; k, n = 1, 2, \cdots\}$ is a neighbourhood basis at 0 which is countable since β is countable. Hence E is metrizable in the strong topology.

5. Weak Topology

If (E, L, M) is a g.s.i.p. algebra of type (p), for each $T \in L$ and each $u \in M$, we define a linear functional $\phi(x; T, u) = [Tx, u]$ on E. Let F_0 be the family of all such linear functionals. Note that, in general, F_0 is not a vector space. Denote by F the vector space (over the same field as that of E) spanned by F_0.

Proposition 5.1. *E and F constitute a dual pair.*

Proof. If $\phi(x) = 0, \forall \phi \in F$, then $[Tx, u] = 0$, $\forall u \in M$ and $T \in L$. But then $x = 0$. (cf. [10]) Conversely, if for a given $\phi_0 \in F$, we have that $\phi_0(x) = 0$, $\forall x \in E$, then ϕ_0 is the zero element of F.

Notation 5.2. We write $\langle x, \phi \rangle = \phi(x)$, $x \in E, \phi \in F$.
clearly $\langle x, \phi \rangle$ is a bilinear functional on E and F.

Proposition 5.3. *Each $\phi \in F$ is continuous on E in the strong topology.*

Proof. For arbitrary $\varepsilon > 0$ we have that

$$\begin{aligned} |\phi(x;T,u) - \phi(x_0;T,u)| &= |[Tx,u] - [Tx_0,u]| \\ &= |[T(x-x_0),u]| \\ &\leq [T(x-x_0),T(x-x_0)]^{\frac{1}{p}}[u,u]^{\frac{p-1}{p}} < \varepsilon \end{aligned}$$

whenever

$$[T(x-x_0),T(x-x_0)]^{\frac{1}{p}} < \frac{\varepsilon}{[u,u]^{\frac{p-1}{p}}}.$$

Thus, each $\phi \in F$ is a continuous linear functional on E equipped with the strong topology. Hence the continuity of any $\phi \in F$ follows.

Definition 5.4. The coarsest topology on E for which all the linear functional from F are continuous is called the weak topology. The family of all subsets of E $W(x;\phi_1,\cdots,\phi_n) = \{y \in E : |\phi_k(y-x)| < 1,\ k = 1,2,\cdots,n\}$, $\forall \phi_1,\phi_2,\cdots,\phi_n \in F$, $n = 1,2,\cdots$ is a neighbourhood basis at x. Since F_0 generates F, the family of all neighbourhoods $W(0;u_1,T_1,\cdots,u_n,T_n) = \{x \in E : |[T_kx,u_k]| < 1, k = 1,2,\cdots,n\}$ corresponding to all $u_1,\cdots,u_n \in M$, $T_1,\cdots,T_n \in L$, $n = 1,2,\cdot$ is also a neighbourhood basis at 0.

Remark 5.5. Since E and F form a dual pair, then E is a Hausdorff topological space in the weak topology.

Proposition 5.6. *Let (E,L,M) be a g.s.i.p. algebra of type (p). Then (E,L,M) equipped with the weak topology is a locally convex algebra.*

Proof. It follows from the general properties of weak topologies.

Acknowledgement. I wish to express my grateful thanks to Dr. S. M. Khaleelulla and Dr. J. Rizvi for their kind help during the preparation of this paper.

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MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, KING ABDULAZIZ UNIVERSITY, P.O.BOX 9028, JEDDAH, SAUDI ARABIA