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THE FORMAL SOLUTION OF A SYSTEM OF OPERATOR DIFFERENTIAL EQUATIONS

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Introduction

The system of operator differential equation

$$X'(\lambda) = A(\lambda)X(\lambda) + B(\lambda) \tag{1}$$

is considered, where $X(\lambda), B(\lambda)$ are vectors, $A(\lambda)$ is n by n matrix and their elements are operator functions which map the interval $[0, \infty)$ into the field M of Mikusiński operator [1], which contains the differential operator s, the integral operator ℓ and translation operator. So the system (1) contains some classes of partial differential equations, integral equations, difference equations and their combinations.

1. Some Notions and Notations

The class C of continuous complex valued function of a non-negative real variable forms a commutative algebra without zero divisors where the product is defined as the finite convolution ($fg = \{\int_0^t f(t-\tau)g(\tau)d\tau\}$), and the sum and scalar product are defined in the usual way. The quotient field of this algebra is the operator field M of Mikusiński [1]. Let $f(t)$ be a continuous function in $0 \leq t < \infty$, $f = \{f(t)\}$ denote the representation of $f(t)$ in C. The elements of M are "convolution quotients" $\frac{f}{g}$, where f and g are in C, $g \neq 0$. In M we have $\frac{h}{k} = \frac{f}{g}$ if and only if $hg = kf$ (in C), so an element $\frac{f}{g}$ in M is a representative of an equivalence class.

Notation 1. If all the elements of a matrix belong to a set S, we say that *the matrix is over the set S.*

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Notation 2. A diagonal matrix whose diagonal elements in order are $d_1, d_2, \cdots, d_n$ will be denoted by $D[d_1, d_2, \cdots, d_n]$. The class K is the set of all functions of type $f_k = \sum_\nu a_{k\nu} e^{i\nu\lambda}$ such that for any λ in $[0, \infty)$, $|f_k| < \infty$, where $a_{k\nu}$ are numerical coefficients.

2. The system of Differential Equation

B. Stanković [4] proved the unicity of the solution of the system (1) with initial conditions when $A(\lambda)$ is over $C_s(\lambda)$ and $B(\lambda)$ is over $\bar{C}_s(\lambda)$ where C_s and $\bar{C}_s$ are Stanković spaces [4]. In an earliar paper [3] we investigated the formal solution of linear differential equation of order n with coefficients whose variable parts consist of funciton of class $s^{-\beta}K, s^{-\beta} = \ell^\beta = \{\frac{t^{\beta-1}}{\Gamma(\beta)}\}, \beta > 0, 0 \leq t < \infty$, and corresponding characteristic equation has roots with distinct real part. We shall investigate any form of roots of the characteristic equation decided to work out the method in the case of the system

$$X'(\lambda) = [A + s^{-\beta}F(\lambda)]X(\lambda), \quad \beta > 0 \tag{2}$$

where A is n by n matrix over the field of complex number, $F(\lambda)$ is a matrix over K. Consider the system

$$X'(\lambda) = AX(\lambda) \tag{3}$$

for which the corresponding characteristic equations

$$\mathrm{Det}\|rE - A\| = 0, \quad (E; \text{Unit matrix}) \tag{4}$$

has the roots

$$r_k^{\alpha_k} = p_k^{\alpha_k} + iq_k^{\alpha_k}, \quad (k = 1, 2, \cdots, m \leq n)$$

where α_k is the multiplicity of these roots. The solution of equation (3) will be any of the linear combinations of the expressions

$$\lambda^{\beta i}(\exp r_k^{\alpha_k}\lambda), (\beta_i = 0, 1, 2, \cdots, \alpha_k - 1, k = 1, 2, \cdots, m, i = 1, 2, \cdots, m).$$

When copying we shall denote the roots of the characteristic equation by $r_k (k = 1, 2, \cdots, n)$ and the general solution will be in the form

$$x_g = \sum_k J_{g,k} x_k(\lambda), \quad J_{g,k} = \text{constant}(g = 1, 2, \cdots, n) \tag{5}$$

The solution (5) can be represented in the form

$$X = JY$$

where J is a constant matrix, Y is the vector with components $x_k(\lambda)(k = 1,2,\cdots,n)$. Note that the matrix J has inverse matrix because the function $x_k(k = 1,2,\cdots,n)$ can be expressed by x_g.

3. Some Properties of Matrices

The proof of our theorem on the system (2) requires several properties of matrices [2] which will be mentioned without proofs.

Let A be the matrix of the system (3) has the form $D[J^{\alpha_1}, J^{\alpha_2},\cdots,J^{\alpha_m}]$, where J^{α_k} is a constant matrix of order $\alpha_k(k = 1,2,\cdots,m)$, J is the matrix formed by $J_{g,k}$ and $Q = D[q_1,q_2,\cdots,q_n]$ where $q_i(i = 1,2,\cdots,n)$ are imaginary parts of the roots of the equation (4). Then:
(i) JAJ^{-1} has the same form of diagonal matrix of A.
(ii) $QJAJ^{-1} = JAJ^{-1}Q$.
(iii) $e^{iQ\lambda}JAJ^{-1} = JAJ^{-1}e^{iQ\lambda}$
(iv) The characteristic roots of the matrix $A_o = J^{-1}AJ - iQ$, are real.

Lemma. *Referring to property (iv), we can show that the solution of the system*

$$Z'(\lambda) - A_o Z(\lambda) + Z(\lambda)A_o = \sum_{\mu\neq 0} a_\mu e^{i\mu\lambda}, a_\mu = const., \tag{6}$$

has the form

$$Z(\lambda) = \sum_{\mu\neq 0} b_\mu e^{i\mu\lambda}, \quad b_\mu = const., \tag{7}$$

Proof. From (7) and (6) we obtain

$$i\mu b_\mu - A_o b_\mu + b_\mu A_o = a_\mu(\mu \neq 0) \tag{8}$$

then the proof of the lemma follows when we solve the linear non–homogeneous system (8) for b_μ. For this let us assume the system of equation (8), can not be solved. Then the corresponding system of the homogeneous equations

$$i\mu\beta_\mu - A_o\beta_\mu + \beta_\mu A_o = 0 \tag{9}$$

permits a particular solution $\beta_\mu \neq 0$, such a solution will lead to a contradiction.

Let $\theta(\lambda) = \beta_\mu e^{i\mu\lambda}$ and $\sigma = e^{-A_o\lambda}\theta e^{A_o\lambda}$, then from (9) we have

$$\frac{d\theta}{d\lambda} - A_o\theta + \theta A_o = 0$$

and

$$\frac{d\sigma}{d\lambda} = e^{-A_o\lambda}[\frac{d\theta}{d\lambda} - A_o\theta + \theta A_o]e^{A_o\lambda} = 0.$$

This implies that $\sigma =$ constant. Let u and v be two constant vectors such that

$$(u\theta(\lambda)v) = \sum_{j,k} u_j\theta_{j,k}(\lambda)v_k,$$

which can be written as

$$(u\theta(\lambda)v) = ue^{A_o\lambda}\sigma e^{-A_o\lambda}v = e^{i\mu\lambda}(u\beta_\mu v) = \phi(\lambda)\sigma\psi(\lambda),$$

where, $\phi(\lambda) = ue^{A_o\lambda}$ and $\psi(\lambda) = ve^{-A_o\lambda}$. But $\phi'(\lambda) = \phi(\lambda)A_o$, $\psi'(\lambda) = -A_o\psi(\lambda)$, and the roots of the equation Det $\|rE - A_o\| = 0$ are $r(k) = p_k^{\alpha_k}(k = 1, 2, \cdots, m)$. Then $\phi(\lambda)$ is the linear combination of $\sum_k \eta_k(\lambda)\exp r(k)\lambda (k = 1, 2, \cdots, m)$, where η_k are polynomials in λ. By similar consideration $\psi(\lambda)$ is the linear combination of $\sum_k \xi_k(\lambda)\exp(r_k)\lambda$ $(k = 1, 2, \cdots, m)$, where $\xi_k(\lambda)$ are polynomials in λ. Consequently, $\phi(\lambda)\sigma\psi(\lambda)$ must be the product of the polynomials of real exponents, it can not be equal to the constant magnitude multiplied by $e^{i\mu\lambda}(\mu \neq 0)$. We have come to the contradiction which establishes the solvability of the system.

4. Form of the Formal Solution of the System (2)

In studing the form of the formal solution of equation (2) we shall prove the following theorem.

Theorem. *The formal solution of the equation (2) is the vector*

$$X = [Je^{iQ\lambda} + \sum_{n=1}^{\infty} s^{-n\beta}P_n(\lambda)]y \quad \beta > 0 \tag{10}$$

where $P_n(\lambda)(n = 1, 2, \cdots)$ are matrices over K, y is the formal solution of the equation

$$\frac{dy}{d\lambda} = \sum_{k=0}^{\infty}(s^{-k\beta}A_k)y \tag{11}$$

where $A_k(k = 0, 1, 2, \cdots,)$ are constant matrices.

Proof. The proof of the theorem follows from the possibility of the formation of matrices $A_k(k=0,1,2,\cdots)$ and $P_n(\lambda)(n=1,2,\cdots)$ which we are just going to formulate and prove.

Substituting (10) in (2), we get

$$[Je^{iQ\lambda}+\sum_{n=1}^{\infty}s^{-\beta n}P_n(\lambda)]y'+[JiQe^{iQ\lambda}+\sum_{n=1}^{\infty}s^{-\beta n}P_n'(\lambda)]y$$

$$\equiv[A+s^{-\beta}F(\lambda)]\cdot[Je^{iQ\lambda}+\sum_{n=1}^{\infty}s^{-\beta n}P_n(\lambda)]y$$

or taking into account (11),

$$[Je^{iQ\lambda}+\sum_{n=1}^{\infty}s^{-\beta n}P_n(\lambda)](\sum_{k=0}^{\infty}s^{-\beta k}A_k)+[JiQe^{iQ\lambda}+\sum_{n=1}^{\infty}s^{-\beta n}P_n'(\lambda)] \quad (12)$$

$$\equiv[A+s^{-\beta}F(\lambda)](Je^{iQ\lambda}+\sum_{n=1}^{\infty}s^{-\beta n}P_n(\lambda)).$$

Let us equate the coefficients of the different power of $s^{-\beta}$:

1) $(s^{-\beta})^o: Je^{iQ\lambda}A_o+JiQe^{iQ\lambda}=AJe^{iQ\lambda}$.

Since J has the inverse matrix J^{-1} and from, the property (iii), A_o can be expressed as

$$A_o=J^{-1}AJ-iQ. \quad (13)$$

2) $(s^{-\beta}): Je^{iQ\lambda}A_1+P_1(\lambda)A_o+P_1'(\lambda)=F(\lambda)Je^{iQ\lambda}+AP_1(\lambda)$,

or

$$P_1'(\lambda)-AP_1(\lambda)+P_1(\lambda)A_o=F(\lambda)Je^{iQ\lambda}-Je^{iQ\lambda}A_1. \quad (14)$$

But the r.h.s. of equation (14) is over K. We put

$$P_1(\lambda)=Je^{iQ\lambda}Z_1(\lambda) \quad (15)$$

Then from (14) and in view of (13) we can show that

$$Z_1'(\lambda)-A_oZ_1(\lambda)+Z_1(\lambda)A_o=e^{-iQ\lambda}J^{-1}FJe^{iQ\lambda}-A_1. \quad (16)$$

Since the matrix $e^{-iQ\lambda}J^{-1}FJe^{iQ\lambda}$ is over K, thus it can be written as

$$\sum_{\mu}a_{\mu}^{(1)}e^{i\mu\lambda},\ \text{taking } A_1=a_o^{(1)}.$$

Hence from the given lemma we can show that the solution of (16) has the form $Z_1=\sum_{\mu\neq 0}b_{\mu}^{(1)}e^{i\mu\lambda}$ and the corresponding $P_1(\lambda)$ will be over K.

3)

$$(s^{-\beta})^2 : P_2'(\lambda) - AP_2(\lambda) + P_1(\lambda)A_1 + P_2(\lambda)A_o = F(\lambda)P_1(\lambda) - Je^{iQ\lambda}A_2. \quad (17)$$

But the r.h.s. of equation (17) is over K. Hence

$$P_2(\lambda) = Je^{iQ\lambda}Z_2(\lambda)$$

Taking into account (13) then (17) will be reduced to

$$Z_2'(\lambda) - A_oZ_2(\lambda) + Z_2(\lambda)A_o = e^{-iQ\lambda}J^{-1}FJe^{iQ\lambda}Z_1(\lambda) - Z_1(\lambda)A_1 - A_2. \quad (18)$$

Since the r.h.s. of equation (18) is over K, it can be represented by $\sum_\mu a_\mu^{(2)} e^{i\mu\lambda}$.

$$Z_2(\lambda) = \sum_{\mu \neq 0} b_\mu^{(2)} e^{i\mu\lambda}, \quad b_\mu^{(2)} = \text{constant}$$

and choosing the quantity $A_2 = a_o^{(2)}$, we obtain from (18)

$$i\mu b_\mu^{(2)} - A_o b_\mu^{(2)} + b_\mu^{(2)} A_o = b_\mu^{(2)} \quad (\mu \neq 0) \quad (19)$$

The solvability of the system (19) is established on the basis of the given lemma as well as the possibility of the formation of matrices A_2 and $P_2(\lambda)$.

4)

$$\begin{aligned}(s^{-\beta})^n : P_n'(\lambda) - AP_n(\lambda) + P_{n-1}(\lambda)A_1 + P_{n-2}(\lambda)A_2 + \cdots \\ +P_1(\lambda)A_{n-1} + P_n(\lambda)A_o = F(\lambda)P_{n-1}(\lambda) - Je^{iQ\lambda}A_n.\end{aligned} \quad (20)$$

The r.h.s of (20) is over K.

Putting $P_n(\lambda) = Je^{iQ\lambda}Z_n(\lambda)$. From (13) equation (20) take the form

$$\begin{aligned}Z_n'(\lambda) - A_oZ_n(\lambda) + Z_n(\lambda)A_o = e^{iQ\lambda}J^{-1}F(\lambda)Je^{iQ\lambda}Z_{n-1} \\ -Z_{n-1}(\lambda)A_1 - Z_{n-2}(\lambda)A_2 - \cdots - Z_1(\lambda)A_{n-1} - A_n.\end{aligned} \quad (21)$$

Also we can show that the r.h.s. of equation (21) can be represented in the form $\sum_{\mu\neq 0} a^{(n)}e^{i\mu\lambda}$ where $A_n = a_0^{(n)}, a_o^{(n)}$ is the free member of $\sum_{\mu=0} a^{(n)}e^{i\mu\lambda}$. Putting the solution of equation (21) in the form $Z_n = \sum_{\mu\neq 0} b_\mu^{(n)}e^{i\mu\lambda}$, $b_\mu^{(n)} = \text{constant}$. We have

$$i\mu b_\mu^{(n)} - A_o b_\mu^{(n)} + b_\mu^{(n)} A_o = a_\mu^{(n)} \quad (\mu \neq 0). \quad (22)$$

The solvability of the system (22) is established on the same basis of the argument discussed for the above lemma. This establishes the solvability of the system (22) as well as the possibility of the formation of matrices A_n and P_n. The process of determining A_n and P_n can be indefinitly extended, which establishes the existance of a formal solution of form (10) of equation (2).

However, the problem of the property of series $\{\sum_{n=1}^{\infty} s^{-\beta n} P_n(\lambda)\}y$ and $\{\sum_{k=0}^{\infty} s^{-\beta k} A_k\}y$ remain open question.

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