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ON GENERALIZED FLOQUET THEORY

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1. Introduction

One basic theory for the linear periodic system

$$\dot{x} = A(t)x, \quad -\infty < t < \infty, \tag{1.1}$$

is, of course, that of Floquet's which says that the solution of the system (1.1) with $A(t+\omega) = A(t)$ in the form of the fundamental matrix $\Phi(t)$ can be expressed as

$$\Phi(t) = p(t)\exp(tR), \quad p(t+\omega) = p(t)$$

and R is a constant matrix

We shall use the following definition introduced in [1].

Definition 1. If f denote a function on some interval I into itself it will be convenient to denote by $f^{[n]}$, for every non-negative integer n, the functions defined inductively by $f^{[0]}(t) = t$ and $f^{[n]}(t) = f(f^{[n-1]}(t))$ for $n > 0$ and $t \in I$.

Definition 2. If f denotes a function on I into itself then any function (or matrix function) P is f-periodic in I if $P(f(t)) = P(t)$ for all t in I.

Definition 3. The system (1.1) is a Generalized Floquet system with respect to f (or GFS-f) if f is an absolutely continuous function on (α, ∞), $\alpha \geq -\infty$, such that

$$f'(t)A(f(t)) = A(t), \tag{1.2}$$

for almost all $t \geq \alpha$ and

$$f(t) > t, \tag{1.3}$$

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for all $t > \alpha$

Clearly the case A periodic with period ω is given by (1.2) with $f(t) = t + \omega$. In what follows it will be assumed without loss of generality that α is negative.

2. Main Results

Let X be the principal matrix solution of a GFS–f (1.1). It is a well known result of a generalized Floquet theory ([1], page 189) that (1.2) implies

$$X(f^{[n]}(t)) = X(t)V^n \tag{2.1}$$

for every $t \in I$, V is a constant nonsingular matrix, $V = X(f(0))$, and if $t_n = f^{[n]}(0)$ then $V = X(t_1), t_1, t_n \in I$.

Theorem 2.1. *Assume that the system (1.1) is GFS-f and the following conditions are satisfied;*

(i) *There exists a matrix B which is given a.e. (almost everywhere) on an interval I by*

$$\Phi_i'\Phi_i^{-1} + f_i'\Phi_i A(f_i)\Phi_i^{-1} = B, \quad i = 1, 2, \tag{2.2}$$

where Φ_1 and Φ_2 are nonsingular matrices, f_1 and f_2 are real-valued functions on the interval I into (α, ∞), the entries of Φ_i and the functions f_i being absolutely continuous on I and the prime denote derivatives.

(ii) *There exists points t_1 and t_2 in I such that*

$$f_i(t_i) = f^{[m_i]}(f_j(t_i)), \quad i = 1, 2, i \neq j \tag{2.3}$$

where $m_1 = 0$ and $m_2 = \pm 1$, and

$$\Phi_i(t_i) = c\Phi_2(t_i), \quad i = 1, 2 \tag{2.4}$$

where c is a scalar constant.

Then every solution of (1.1) is f–periodic on (α, ∞).

Corollary 2.1. *If the system (1.1) is a GFS–f and there exist absolutely continuous functions f_1 and f_2 on some interval I into $(\alpha, \infty), \alpha \geq -\infty$ such that*

$$f_1'A(f_1) = f_2'A(f_2) \qquad \text{a.e. in I} \tag{2.2*}$$

and if there exist points t_1 and t_2 in I at which (2.3) holds then every solution of (1.1) is f-periodic on (α, ∞).

Proof. The result follows by taking $\Phi_1 = \Phi_2 = U$ (the unit matrix).

Proof of Theorem 2.1

Let $Y_i = \Phi_i X(f_i), i = 1,2$ where X is a fundamental matrix solution of (1.1) as in [6], Y_1 and Y_2 are nonsingular since Φ_i and X are nonsingular, so that Y_1 and Y_2 are fundamental solution matrices of

$$y' = B(t)y. \tag{2.5}$$

Consider

$$\begin{aligned} Y_i(t_i) &= \Phi_i(t_i)X(f_i(t_i)) \\ &= \Phi_i(t_i)Xf^{[m_i]}(f_j(t_i))) \end{aligned}$$

by using (2.3). Hence by using (2.1) we have

$$Y_i(t_i) = \Phi_i(t_i)\Phi_j^{-1}(t_i)Y_j(t_i)V^{m_i},$$

so that for $m_1 = 0$ we have

$$Y_1(t_1) = \Phi_1(t_1)\Phi_2^{-1}(t_1)Y_2(t_1).$$

Thus by using (2.4) we have

$$\begin{aligned} Y_1(t_1) &= c\Phi_2(t_1)\Phi_2^{-1}(t_1)Y_2(t_1) \\ &= cY_2(t_1). \end{aligned}$$

Similary

$$Y_2(t_2) = \Phi_2(t_2)\Phi_1^{-1}(t_2)Y_1(t_2)V^{m_2}.$$

Hence by using (2.4) we have

$$Y_2(t_2) = (1/c)Y_1(t_2)V^{m_2}.$$

We note that the piecewise continuity of A and absolute continuity of Φ_i and f_i are sufficient conditions for the uniqueness of solutions of (2.5), and hence

$$Y_1 = cY_2 \quad \text{and} \quad Y_2 = (1/c)Y_1V^{m_2} \quad \text{everywhere in } I.$$

Thus $Y_2 = Y_2V^{m_2}$ and $V^{m_2} = U$ since Y_2 is nonsingular and since $m_2 = \pm 1$, then $V = U$ and by using (2.1) it follows that

$$X(f^{[n]}(t)) = X(t)$$

for every $t > \alpha$, and integer n. This completes the proof.

Theorem 2.2. *Assume that there exists a continuous or piecewise continuous matrix B such that the following condituons are satisfied:*

(i) *A is given a.e. on two intervals J_1 and J_2,*

$$(J_i \subset (\alpha, \infty), \alpha > -\infty, i = 1, 2) \quad by$$

$$\Psi_i' \Psi_i^{-1} + g_i' \Psi_i B(g_i) \Psi_i^{-1} = A, \quad i = 1, 2, \tag{2.6}$$

where Ψ_1 and Ψ_2 are non-singular matrices, g_1 and g_2 are real-valued functions on J_1 and J_2 respectively, the intries of Ψ_i and g_i being absolutely continuous on J_i.

(ii) *There is a point τ_i in J_i such that $f^{[n_i]}(\tau_i) \in J_j$*

$$g_i(\tau_i) = g_j(f^{[n_i]}(\tau_i)), \quad i = 1, 2, i \neq j \tag{2.7}$$

where $n_1 = 0$ and $n_2 = \pm 1$;

$$\Psi_1(f^{[n_i]}(\tau_i)) = k \Psi_2(\tau_i), \quad i = 1, 2 \tag{2.8}$$

where k is a scalar constant.

If the system (1.1) is GFS–f then every solution of it is f–periodic on (α, ∞).

The case $\Psi_1 = \Psi_2 = U$ (the unit matrix) may be stated as the following corollary:

Corollary 2.2. *If the system (1.1) is GFS-f and there is a piecewise continuous matrix B and absolutely continuous functions g_1 and g_2 on intervals J_1 and J_2 respectively where $J_i \subset (\alpha, \infty), \alpha > -\infty, i = 1, 2,$ such that*

$$g_i' B(g_i) = A, \quad a.e. \ in \ J_i, \quad i = 1, 2$$

and if (2.7) holds at points τ_i in J_i, then every solution of (1.1) is f–periodic on (α, ∞).

Proof of theorem 2.2. Let Y be a fundemental solution matrix of (2.5) and let $X_i = \Psi_i Y(g_i)$, then as in the proof of theorem 2.1, one finds that X_1 and X_2 are fundamental solution matrices of (1.1) on J_1 and J_2 respectively. Consider

$$X_i(\tau_i) = \Psi_i(\tau_i) Y(g_i(\tau_i)).$$

Then by using (2.7) we have

$$\begin{aligned} X_i(\tau_i) &= \Psi_i(\tau_i)Y(g_i(f^{[n_i]}(\tau_i))) \\ &= \Psi_i(\tau_i)\Psi_j^{-1}(f^{[n_i]}(\tau_i))X_j(f^{[n_i]}(\tau_i)). \end{aligned}$$

Hence by using (2.1) it follows that

$$X_i(\tau_i) = \Psi_i(\tau_i)\Psi_j^{-1}(f^{[n_i]}(\tau_i))X_j(\tau_i)V^{n_i},$$

so that

$$\begin{aligned} X_1(\tau_1) &= \Psi_1(\tau_1)\Psi_2^{-1}(\tau_1)X_2(\tau_1) \quad \text{for} \quad n_1 = 0 \\ &= k\Psi_2(\tau_1)\Psi_2^{-1}(\tau_1)X_2(\tau_1) \\ &= kX_2(\tau_1), \end{aligned}$$

by using (2.8). Also, we have

$$\begin{aligned} X_2(\tau_2) &= \Psi_2(\tau_2)\Psi_1^{-1}(f^{[n_2]}(\tau_2))X_1(\tau_2)V^{n_2} \\ &= (1/k)\Psi_1(f^{[n_2]}(\tau_2)\Psi_1^{-1}(f^{[n_2]}(\tau_2))X_1(\tau_2)V^{n_2} \\ &= (1/k)X_1(\tau_2)V^{n_2}, \end{aligned}$$

by using (2.8). Thus as in the proof of theorem 2.1, $V = U$, and X is f-periodic on (α, ∞). This completes the proof.

Remark 1. In the case that either f_1 and f_2 or g_1 and g_2 are monotonic, Theorems 2.1 and 2.2 are statements of the same results, in this case also corollaries 2.1 & 2.2 are equivalent. For example, if f_1 and f_2 are monotonic we will show that the conditions (2.2), (2.3) and (2.4) of Theorem 2.1, may be written in the form of (2.6), (2.7) and (2.8) of Theorem 2.2, respectively.

Let $J_i = f_i(I)$ and define $g_i = f_i^{-1}$ (the inverse funciton of f_i) and $\Psi_i = \Phi_i^{-1}(g_i), (\Phi_i^{-1}$ is the multiplicative inverse of $\Phi_i)$. Then as in [5], equation (2.2) may be written in the form of (2.6). If we take $\tau_i = f_i(t_i)$ then (2.3) becomes

$$\tau_i = f^{[m_i]}(f_j(g_i(\tau_i)))$$

i.e.

$$f_j^{-1}(f^{[-m_i]}(\tau_i)) = g_i(\tau_i)$$

where $f^{[-m_i]}$ is the inverse of $f^{[m_i]}$ and this exists for the sequence $f^{[m_i]}$ is increasing sequence from (1.3). Thus

$$g_i(f^{[n_i]}(\tau_i)) = g_i(\tau_i), \quad n_i = -m_i$$

which is (2.7). Also (2.4) becomes

$$\begin{aligned}(c\Phi_2)^{-1}(t_i) &= \Phi_1^{-1}(t_i)\\ (1/c)\Phi_2^{-1}(t_i) &= \Phi_1^{-1}(t_i)\\ (1/c)\Phi_2^{-1}(g_i(\tau_i)) &= \Phi_i^{-1}(g_i(\tau_i)), i = 1,2.\end{aligned}$$

For $i = 1$, we have

$$\begin{aligned}(1/c)\Phi_2^{-1}(g_1(\tau_1)) &= \Phi_1^{-1}(g_1(\tau_1))\\ (1/c)\Phi_2^{-1}(g_2(\tau_1)) &= \Psi_1(\tau_1)\end{aligned}$$

by using (2.7) for $n_1 = 0$. Hence

$$(1/c)\Psi_2(\tau_1) = \Psi_1(\tau_1).$$

Thus

$$\Psi_1(f^{[n_1]}(\tau_1)) = k\Psi_2(\tau_1), \quad k = 1/c. \tag{2.9}$$

For $i = 2$, we have

$$\begin{aligned}(1/c)\Phi_2^{-1}(g_2(\tau_2)) &= \Phi_1^{-1}(g_2(\tau_2))\\ (1/c)\Psi_2(\tau_2) &= \Phi_1^{-1}(g_1(f^{[n_2]}(\tau_2))\end{aligned}$$

by using (2.7). Hence

$$k\Psi_2(\tau_2) = \Psi_1(f^{[n_2]}(\tau_2)) \tag{2.10}$$

From (2.9) and (2.10), we have

$$\Psi_1(f^{[n_i]}(\tau_i)) = k\Psi_2(\tau_i), \quad i = 1,2,$$

which is (2.8).

Remark 2. The result of [6] is obtained by taking $f(t) = t + \omega$.

Remark 3. The result in [4] (Theorem 2, pp. 691) is obtained by corollary 2.1 with $f(t) = t + \omega, f_1(t) = t, f_2(t) = -t, t_1 = 0$ and $t_2 = -\omega/2$, so that $m_1 = 0$, $m_2 = +1$.

By the generalized Floquet theory [1], if X_0 is any continuous nonsingular matrix on (α, ∞) such that $X_0(0) = U$ and satisfying (2.1), then there exists a continuous nonsingular matrix P which is f-periodic on (α, ∞) such that $P(0) = U$ and

$$X(t) = P(t)X_0(t) \tag{2.11}$$

and (as in [5], pp. 19–20), there is at least one solution $x(t)$ of (1.1) such that

$$x(f(t)) = \lambda x(t) \tag{2.12}$$

for all t, where $\lambda \neq 0$ is a constant (real or complex). In fact if $x(t)$ is a solution of (1.1), then there exists a constant vector x_0 such that

$$x(t) = X(t)x_0 = P(t)X_0(t)x_0 \tag{2.13}$$

by using (2.9). If $x(t)$ is to satisfy (2.10), then

$$P(f(t))X_0(f(t))x_0 = \lambda P(t)X_0(t)x_0$$

$$P(t)X_0(t)Vx_0 = \lambda P(t)X_0(t)x_0$$

by using (2.1). Hence

$$P(t)X_0(t)(V - \lambda U)x_0 = 0.$$

Since P and X_0 are nonsingular then

$$(V - \lambda U)x_0 = 0. \tag{2.14}$$

Consequently if λ is an eigenvalue of V and x_0 is a corresponding eigen–vector, then the solution $x(t)$ defined by (2.13) has the desired property. Thus we have:

Theorem 2.3. *If the system (1.1) is GFS-f, then it has a f–periodic solution if and only if there exists an eigenvalue of V which is equal to* 1. *Also if there is an eigenvalue of V which is equal to 1, then the system (1.1) has a $f^{[2]}$–periodic solution.*

Proof. Since

$$x(f^{[2]}(t)) = x(f(f(t)) = -x(f(t)) = x(t)$$

the result follows by using (2.10) with $\lambda = -1$.

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