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ON THE UNICELLULARITY OF VOLTERRA–TYPE INTEGRAL OPERATORS

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In this paper we are going to study the unicellularity of Volterra–type operators. We know the Volterra operator is unicellular, and there are several approaches to prove it, for example, in [1] and [5]. One of the approaches is using the Titchmarsh convolution theorem, a simple proof of which can be found in [4]. The unicellularity of the Volterra operator and the Titchmarsh convolution theorem are proved independently. But in [3], Kalish showed that the above two statements are equivalent. We will consider a certain kind of Volterra–type operator and the generalized Titchmarsh convolution theorem. We will show the unicellularity of this Volterra–type operator by using the Titchmarsh convolution theorem and then prove the generalized Titchmarsh convolution theorem using the unicellularity.

Definition 1 [5]: A Volterra–type integral operator on $L^2[0,1]$ is an operator A of the form $(Af)(x) = \int_0^x K(x,t)f(t)dt$, where K is any square–integrable (with respect to area measure) function on the unit square. K is called its kernel. The Volterra operator V is obtained when the kernel K is the constant function 1 on the unit square. Explicitly, the Volterra operator is defined on $L^2[0,1]$ by $(Vf)(x) = \int_0^x f(t)dt$. In particular, we want to consider a kind of Volterra–type integral operator on $L^2[0,1]$ by giving two functions in $L^2[0,1]$.

Definition 2. Let $w(t)$ and $q(t)$ be in $L^2[0,1]$. We denote the Volterra–type operator defined by $(V_{q,w}f)(x) = q(x)\int_0^x f(t)w(t)dt$ for $f \in L^2[0,1]$ by $V_{q,w}$. The kernel is $K(x,t) = \chi_{[0,x]}(t)q(x)w(t)$ on the unit square.

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Theorem 3. *For a Volterra type integral operator* $V_{q,w}$, *let* $Q(t) = \int_0^t q(s) w(s)ds$, *then* $(V_{q,w}^n)(x) = (1/(n-1)!)q(x)\int_0^x (Q(x)-Q(t))^{n-1}f(t)w(t)dt$. *The kernel is* $K_{q,w}^n(x,t) = (1/(n-1)!)q(x)\chi_{[0,x]}(t)(Q(x)-Q(t))^{n-1}w(t)$. *(Here,* $V_{q,w}^n$ *means the n–th power of* $V_{q,w}$ *and* $K_{q,w}^n$ *denotes the Volterra kernel of the operator* $V_{q,w}^n$*).*

Proof. We proceed by induction on n. For $n=1$, $(V_{q,w}f)(x) = q(x)\int_0^x f(t) w(t)dt$, i.e., $K_{q,w}(x,t) = q(x)\chi_{[0,x]}(t)w(t)$. We assume $K_{q,w}^n(x,t) = (1/(n-1)!)q(x)\chi_{[0,x]}(t)(Q(x)-Q(t))^{n-1}w(t)$. Then by using integration by parts.

$$\begin{aligned}
(V_{q,w}^{n+1}f)(x) &= (V_{q,w}^n V_{q,w}f)(x) \\
&= (1/(n-1)!)q(x)\int_0^x (Q(x)-Q(t))^{n-1}(V_{q,w}f)(t)w(t)dt \\
&= (1/(n-1)!)q(x)\int_0^x (Q(x)-Q(t))^{n-1}q(t)w(t) \\
&\quad [\int_0^t f(s)w(s)ds]dt \\
&= (1/(n-1)!)q(x)[(-1/n)(Q(x)-Q(t))^n\int_0^t f(s)w(s)ds \\
&\quad + \int_0^x (1/n)(Q(x)-Q(t))^n f(t)w(t)dt] \\
&= (1/n!)q(x)\int_0^x (Q(x)-Q(t))^n f(t)w(t)dt.
\end{aligned}$$

Hence, $K_{q,w}^n(x,t) = (1/n!)q(x)\chi_{[0,1]}(t)(Q(x)-Q(t))^n w(t)$.

For $0 \le a \le b \le 1$, $L^2[a,b]$ means the closed subspace of $L^2[0,1]$ consisting of functions vanishing a.e. on the complement of $[a,b]$. Let f be in $L^2[0,1]$. The support of f is the complement of the largest open subset of $[0,1]$ space where $f=0$ a.e.. It is denoted by supp f. For each a $\in [0,1]$, $L^2[a,1]$ is invariant under every Volterra–type integral operator A. A well–known fact is that every invariant subspace of $L^2[0,1]$ under the Volterra operator V is one of the $L^2[a,1], 0 \le a \le 1$; i.e. V is unicellular.

Now we will consider the unicellularity of $V_{q,w}$. The proofs follow a strategy first used by Kalish in [3] for the Volterra operator V. We prove a lemma that transforms a problem about invariant subspaces to a problem of cyclic vectors.

Lemma 4. *If the only invariant subspaces for* $V_{q,w}$ *are* $L^2[a,1]$, $0 \le a \le 1$, *then* f *is cyclic for* $V_{q,w}$ *whenever* $0 \in$ *supp* f.

Proof. Assume that the only invariant subspaces for $V_{q,w}$ are $L^2[a,1], 0 \leq a \leq 1$. If $f \in L^2[0,1]$ and $0 \in$ supp f, then span$\{f, V_{q,w}f, \cdots\} = L^2[0,1]$, since the subspace span$\{f, V_{q,w}f, \cdots\}$ is invariant under $V_{q,w}$.

Corollary 5. *$V_{q,w}$ is unicellular, then f is cyclic for $V_{q,w}$, whenever $0 \in$ supp f.*

We will introduce some notation. $[V_{q,w}]$ stands for the statement that the only closed $V_{q,w}$-invariant subspaces of $L^2[0,1]$ are the spaces $L^2[a,1]$, $0 \leq a \leq 1$. That is, $[V_{q,w}]$ denotes the statement of the unicellularity of $V_{q,w}$. $[T_{q,w}]$ stands for the generalized Titchmarsh Convolution theorem: If f and g are in $L^2[0,1]$, 0 is in supp f, and if $f \otimes g = 0$ a.e. on $[0,1]$, then $g = 0$ a.e. on $[0,1]$, where $f \otimes g = q(x) \int_0^x f(x-t)g(t)w(t)dt$. In case $q(t)$ and $w(t)$ are both equal to the constant function 1 on $[0,1]$, $[T]$ will denote the usual Titchmarsh Convolution theorem given by substituting $L^1[0,1]$ instead of $L^2[0,1]$ in $[T_{q,w}]$.

Theorem 6. *If $w(t)$ is non-vanishing a.e. on $[0,1]$, and $q(t)$ is continuous on $[0,1]$ and non-vanishing on $[0,1]$, then $[V_{q,w}]$ implies $[T_{q,w}]$.*

Proof. Assume that f and g are in $L^2[0,1]$, $f \otimes g = 0$, and $0 \in$ supp f. Then by Lemma 4, f is cyclic for $V_{q,w}$.

Case 1. f and g are continuous on $[0,1]$: From the assumption, $f \otimes g = 0$. If e is the constant function 1 on $[0,1]$, then $V_{q,w}f = e \otimes f$ and $(V_{q,w}^n f) \otimes g = (e^n \otimes f) \otimes g = e^n \otimes (f \otimes g) = 0$ for all n. So, $q(x) \int_0^x (V_{q,w}^n f)(t) g(x-t)w(t)dt = 0$ in $[0,1]$ for all n. Let $x = 1$. Then

$$\begin{aligned} 0 &= q(1) \int_0^1 (V_{q,w}^n f)(t) g(1-t) w(t) dt \\ &= \langle (V_{q,w}^n f)(t), \overline{q(1)g(1-t)w(t)} >_{L^2[0,1]} . \end{aligned}$$

So, $\overline{q(1)g(1-t)w(t)} \perp (V_{q,w}^n f)(t)$ for all n. Since f is cyclic for $V_{q,w}, q(1)$ $g(1-t)w(t) = 0$ a.e. on $[0,1]$. But $q(t) \neq 0$ and $w(t)$ is non-vanishing a.e. on $[0,1]$. Hence $g(t) = 0$ a.e. on $[0,1]$. $[T_{q,w}]$ is true for all continuous functions f and g.

Case 2. f and g are in $L^2[0,1]$: If $f \otimes g = 0$ on $[0,1]$, then $0 = e \otimes e \otimes f \otimes g$. So, $(e \otimes f) \otimes (e \otimes g) = 0$ a.e. on $[0,1]$. But $e \otimes f$ and $e \otimes f$ are continuous and $0 \in$ supp $(e \otimes f)$, since $w(t)$ is non-vanishing a.e. By case 1, $e \otimes g = 0$ a.e. on $[0,1]$. Since $q(t)$ is non-vanishing on $[0,1]$ and $w(t)$ is non-vanishing a.e. on $[0,1]$, $g(t) = 0$ a.e. on $[0,1]$.

Theorem 7. *If $q(t)$ and $w(t)$ are positive and continuous on $[0,1]$, then*

f is cyclic for $V_{q,w}$ whenever $0 \in supp\ f$.

Proof. Since $V_{q,w}$ is quasi–nilpotent [2], $(1-\alpha V_{q,w})^{-1}$ exists for all $\alpha \neq 0$. By Theorem 3,

$$
\begin{aligned}
&[(1-\alpha V_{q,w})^{-1}](x)\\
&= \sum_{n=0}^{\infty} \alpha^n (V_{q,w}^n f)(x)\\
&= f(x) + \alpha \sum_{n=1}^{\infty} \alpha^{n-1}(1/(n-1)!)q(x)\int_0^x (Q(x)-Q(t))^{n-1} f(t)w(t)dt\\
&= f(x) + \alpha q(x)\int_0^x e^{\alpha(Q(x)-Q(t))} f(t)w(t)dt.
\end{aligned}
$$

Now we assume $0 \in$ supp f for $f \in L^2[0,1]$. We want to show f is cyclic for $V_{q,w}$ on $L^2[0,1]$. Suppose $\bar{g} \perp V_{q,w}^n f$ for all $n = 0,1,2,\cdots$. Then $\bar{g} \perp (1-\alpha V_{q,w})^{-1} f$ for all $\alpha \neq 0$. For all $\alpha \neq 0$,

$$
\begin{aligned}
0 &= \langle (1-\alpha V_{q,w})^{-1} f, \bar{g} \rangle_{L^2[0,1]}\\
&= \int_0^1 [(1-\alpha V_{q,w})^{-1} f](x) g(x) dx\\
&= \alpha \int_0^1 q(x) (\int_0^x e^{\alpha(Q(x)-Q(t))} f(t)w(t))dt\, g(x)dx.
\end{aligned}
$$

So, $\int_0^1 q(x)\int_0^x e^{\alpha(Q(x)-Q(t))} f(t)w(t)g(x)dtdx = 0$. Let $u(x,t) = Q(x)-Q(t)$. We will change the variables (x,t) to (u,t). Then

$$
\partial(u,t)/\partial(x,t) = \begin{vmatrix} w(x)q(x) & -w(t)q(t) \\ 0 & 1 \end{vmatrix} = w(x)q(x).
$$

Since $w(x)q(x)$ is positive and continous, $Q(t) = \int_0^t w(x)q(x)ds$ is strictly increasing. So, it is invertible. Then

$$
x = Q^{-1}(u + Q(t)),
$$

$$
0 \leq u \leq Q(1),
$$

$$
0 \leq t \leq Q^{-1}(Q(1)-u), \quad \text{and}
$$

$$
\begin{aligned}
0 = &\int_0^{Q(1)} e^{\alpha u} \int_0^{Q^{-1}(Q(1)-u)} f(t)w(t)g(Q^{-1}(u+Q(t)))\\
&\times [1/w(Q^{-1}(u+Q(t)))q(Q^{-1}(u+Q(t)))]dtdu \quad \text{for all} \quad \alpha \neq 0.
\end{aligned}
$$

Hence $0 = \int_0^{Q^{-1}(Q(1)-u)} f(t)w(t)g(Q^{-1}(u+Q(t))(1/w(Q^{-1}(u+Q(t)))dt$ on $0 \le u \le Q(1)$. Let $s = Q^{-1}(Q(1)-u)$. Then $Q(s) = Q(1)-u$ and $0 = \int_0^s f(t)g(Q^{-1}(Q(1)-(Q(s)-Q(t)))w(t)[1/w(Q^{-1}(Q(1)-(Q(s)-Q(t)))]dt$. Let $h(x) = Q^{-1}(x)$ and $k(x) = Q^{-1}(Q(1)-x)$ for $0 \le x \le Q(1)$. Let $F = (f/q) \circ h$ and $G = (g/w) \circ k$ be defined on $0 \le x \le Q(1)$. Then F and G are in $L^1[0, Q(1)]$, $0 \in \text{supp } F$, and $0 = \int_0^s F(Q(t))G(Q(s) - Q(t))q(t)w(t)dt$. Let $Q(t) = v$. Then $w(t)q(t)dt = dv$ and $0 \le v \le Q(s)$. So, $\int_0^{Q(s)} F(v)G(Q(s)-v)dv = 0 = \int_0^{Q(s)} F(Q(s)-v)G(v)dv = F * G$. By the Titchmarsh Convolution theorem $[T]$, (see [4]), $G(v) = 0$ a.e. on $0 \le v \le Q(s)$. But $s = Q^{-1}(Q(1)-u)$ and $0 \le u \le Q(1)$. So, $0 \le Q(s) \le Q(1)$. That is, $(g/w) \circ k(v) = (g/w)(Q^{-1}(Q(1)-v)) = 0$ a.e. on $0 \le v \le Q(1)$. Since w is positive, $g(Q^{-1}(Q(1)-v)) = 0$ a.e. on $0 \le v \le Q(1)$. Hence $g(x) = 0$ a.e. on $[0,1]$. Hence f is cyclic for $V_{q,w}$.

Theorem 8. *If $q(t)$ and $w(t)$ are positive on $[0,1]$ and continuous on $[0,1]$, then $V_{q,w}$ is unicellular.*

Proof. Let M be an invariant subspace of $L^2[0,1]$ for $V_{q,w}$. If M contains f such that $0 \in \text{supp } f$, then f is cyclic for $V_{q,w}$, so $\text{span}\{f, V_{q,w}f, \cdots\} = L^2[0,1] \subset M$, i.e. $M = L^2[0,1]$. Else $0 \notin \text{supp } f$ for any f in M. Thus there is $a_f > 0$ such that $f = 0$ a.e. on $[0, a_f)$ for each $f \in M$. Let $a_f = \sup\{0 < a \le 1 : f = 0 \text{ a.e. on } [0,a)\}$ and if $\alpha = \inf\{a_f : f \in M\}$, then an easy argument shows that $\alpha > 0$. Clearly, $M \subset L^2[\alpha, 1]$, and M contains an element g such that $\alpha \in \text{supp } g$. We will show that g is cyclic for the restriction operator $V_{q,w}|L^2[\alpha,1]$ to conclude that $M = L^2[\alpha,1]$. Let $W(u) = (1-\alpha)^{\frac{1}{2}} w(u(1-\alpha)+\alpha)$ and $Q(u) = (1-\alpha)^{\frac{1}{2}} q(u(1-\alpha)+\alpha)$, for $u \in [0,1]$. Let $U_\alpha : L^2[\alpha,1] \to L^2[0,1]$ be defined by

$$(U_\alpha h)(x) = (1-\alpha)^{\frac{1}{2}} h(x(1-\alpha)+\alpha) \quad \text{for} \quad x \in [0,1].$$

Then it can be easily be shown that U is unitary and that $U(V_{q,w}|L^2[\alpha,1])$ $U^* = V_{Q,W}$. But Q and W are positive and continuous on $[0,1]$, and if $g \in L^2[\alpha,1]$ with $\alpha \in \text{supp } g$, then $U_\alpha g \in L^2[0,1]$ with $0 \in \text{supp } U_\alpha g$. So, by Theorem 7, $U_\alpha g$ is cyclic for $V_{Q,W}$, hence g is cyclic for $V_{q,w}|L^2[\alpha,1]$.

From Theorems 6, 8 and the Titchmarsh Convolution theorem $[T]$ that is proved in [4], we have:

Theorem 9. *If $q(t)$ and $w(t)$ are positive and continuous, then the Volterra-type integral operator $V_{q,w}$ is unicellular. Moreover, the generalized Titchmarsh Convolution theorem is always true.*

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