Youngnam Statistical Letters
August, 1990, Vol. 1, 67-76

Estimation for Functions of Two Parameters in the Pareto Distribution

Jungsoo Woo* and Suk-Bok Kang *

ABSTRACT

For a two-parameter Pareto distribution, the uniformly minimum variance unbiased estimators(UMVUE) for the function of the two parameters are expressed in terms of confluent hypergeometric function. The variance of the UMVUE is also expressed in terms of hypergeometric function of several variables. UMVUE's for the rth moment about zero and several useful parametric functions, and their variances are obtained as special cases. The estimators of Baxter(1980) and Saksena and Johnson(1984) are special cases of our estimator.

1. Introduction

The two-parameter form of the Pareto distribution has the density function given by

$$f(x;\theta,\lambda)=\begin{cases}\theta\lambda^{\theta}/x^{\theta+1}, & \text{if } x\geq\lambda>0,\theta>0\\ 0, & \text{if } x<\lambda\end{cases} \tag{1.1}$$

where positive constants θ and λ are a shape parameter and a scale parameter, respectively. The Pareto distribution has been found to be suitable for approximating the right tails of distribution with positive skewness. It has also been found to adapt to several socio-economic, physical and biological phenomena. Johnson and Kotz(1970) have given a brief description of most of the research work published up to 1970. Likeš(1969) derived the uniformly minimum variance unbiased estimators(UMVUE's) of the two parameters in Pareto distribution, and subsequently a simplified derivation and the variances for the UMVUE's of the two parameters

* *Department of Statistics, Yeungnam University, Gyongsan 713-749, Korea*

were given by Baxter(1980). Saksena and Johnson(1984) showed that the maximum likelihood estimators are jointly complete, and they obtained the best unbiased estimators of the two parameters and the variances of these estimators.

In this paper, we consider a more general class of UMVUE for the function of two parameters in the Pareto distribution, and then obtain the variance of the UMVUE for the function of two parameters. We note that the estimators of Baxter(1980) and Saksena and Johnson(1984) are special cases of our estimator. In section 2 a general result concerning the expectation of a certain class of function of the complete sufficient statistics is proved, and then the UMVUE's of the rth moment of the Pareto and Power-function distributions are obtained as a special cases. Furthermore, UMVUE's are obtained for several other parametric functions. The variances of the UMVUE's for several useful parametric functions are also obtained as a special case.

2. UMVUE for the function of two parameters

A random variable X is distributed as the Pareto with density function (1.1), denoted by $X \sim P(\lambda, \theta)$.

Let the random variables $X_1, X_2, \ldots, X_n$ be independent and all having the distribution(1.1). Epstein and Sobel(1954) proved the following;

Proposition. Let $B = min(X_1, X_2, \ldots, X_n)$ and $A = n/\sum_{i=1}^{n} log(X_i/B)$. Then $B \sim P(\lambda, n\theta), T \equiv 2n\theta/A \sim \chi^2_{2n-2}$, and they are independent.

The sufficient statistic for (λ, θ) is (B, A), and Likeš(1969) derived the UMVUE's of λ and θ in terms of B and A. A simplified proof of the above proposition together with a compact derivation of the estimators was given by Baxter(1980). An alternative proof of the proposition was also given by Malik(1970). The statistic (B, A) is a set of jointly complete sufficient statistic (see Saksena and Johnson(1984)).

Theorem 1. For any real numbers α, β, γ, and δ, and any integer $n \geq 2$ such that $n - \beta - \gamma - 1 > 0$ and $n\theta > \alpha$, the UMVUE of $\lambda^\alpha \theta^\gamma (\theta - \delta)^\beta$ in the Pareto distribution(1.1) is given by

$$U_m = \frac{\Gamma(n-1)}{\Gamma(n-\beta-\gamma-1)} B^\alpha (A/n)^{\beta+\gamma} \Big\{ {}_1F_1(-\beta; n-\beta-\gamma-1; \delta n/A)$$

$$- \frac{\alpha}{(n-\beta-\gamma-1)A} {}_1F_1(-\beta; n-\beta-\gamma; \delta n/A)\Big\} \tag{2.1}$$

where ${}_1F_1(a;b;x)$ is a confluent hypergeometric function.

Proof. A pair of jointly sufficient statistic for (λ, θ) is (B, A), which is also complete, and hence, an unbiased estimator expressed as a function of (B, A) has the uniformly minimum variance. Therefore we shall try to find an unbiased estimator of $\lambda^\alpha \theta^\gamma (\theta - \delta)^\beta$ by $B^\alpha G(A; \theta)$. Since A and B are independent and $B \sim P(\lambda, n\theta)$, the function G is determined by the condition

$$E\big[B^\alpha G(A;\theta)\big] = \frac{n\theta\lambda^\alpha}{n\theta - \alpha} E[G(A;\theta)] = \lambda^\alpha \theta^\gamma (\theta-\delta)^\beta, \quad (n\theta > \alpha).$$

Since $T = 2n\theta/A \sim \chi^2_{2n-2}$,

$$\begin{aligned} E[G(A;\theta)] &= \frac{\theta^{n-1}}{\Gamma(n-1)} \int_0^\infty G(\frac{n}{t};\theta) t^{n-2} e^{-\theta t} dt \\ &= \theta^\gamma (\theta-\delta)^\beta - \frac{\alpha}{n} \theta^{\gamma-1} (\theta-\delta)^\beta. \end{aligned}$$

The function G can be obtained by the inverse Laplace transform; with the aid of 4.18 in Oberhettinger and Badii(1970), the result is

$$\begin{aligned} G(\frac{n}{t};\theta) t^{n-2} =& (n-2)! \Big\{ \frac{1}{\Gamma(n-\beta-\gamma-1)} t^{n-\beta-\gamma-2} {}_1F_1(-\beta; n-\beta-\gamma-1; \delta t) \\ & - \frac{\alpha}{n} \frac{1}{\Gamma(n-\beta-\gamma)} t^{n-\beta-\gamma-1} {}_1F_1(-\beta; n-\beta-\gamma; \delta t) \Big\} \end{aligned}$$

for $n - \beta - \gamma > 1$.

Hence,

$$\begin{aligned} G(A;\theta) =& \frac{\Gamma(n-1)}{\Gamma(n-\beta-\gamma-1)} (n/A)^{-\beta-\gamma} \Big\{ {}_1F_1(-\beta; n-\beta-\gamma-1; \delta n/A) \\ & - \frac{\alpha}{(n-\beta-\gamma-1)A} {}_1F_1(-\beta; n-\beta-\gamma; \delta n/A) \Big\} \end{aligned}$$

for $n - \beta - \gamma > 1$.

This completes the proof.

Next we consider the variance of the UMVUE U_m for the parametric function $\lambda^\alpha\theta^\gamma(\theta-\delta)^\beta$ in the Pareto distribution (1.1).

Theorem 2. For any real numbers α,β,γ, and $\delta>0, n\geq 2$ such that $n-2\beta-2\gamma-1>0$, $n-\beta-\gamma-1>0$, $\theta n>2\alpha$, *and* $\theta\geq 2\delta$, the variance of the UMVUE U_m in (2.1) is given by

$$\begin{aligned}
Var(U_m) = \lambda^{2\alpha}\theta^{2\gamma}\Bigg\{ & \frac{n\Gamma(n-1)\theta^{2\beta+1}\Gamma(n-2\beta-2\gamma-1)}{\{\Gamma(n-\beta-\gamma-1)\}^2(\theta n-2\alpha)} \\
& \Bigg\{F_\delta\Big(n-2\beta-2\gamma-1;-\beta,-\beta;n-\beta-\gamma-1,n-\beta-\gamma-1;\frac{\delta}{\theta},\frac{\delta}{\theta}\Big) \\
& +\frac{\alpha^2(n-2\beta-2\gamma)(n-2\beta-2\gamma-1)}{n^2(n-\beta-\gamma-1)^2\theta^2}F_\delta\Big(n-2\beta-2\gamma+1;-\beta,-\beta; \\
& n-\beta-\gamma,n-\beta-\gamma;\frac{\delta}{\theta},\frac{\delta}{\theta}\Big)-\frac{2\alpha(n-2\beta-2\gamma-1)}{n(n-\beta-\gamma-1)\theta} \\
& F_\delta\Big(n-2\beta-2\gamma;-\beta,-\beta;n-\beta-\gamma-1,n-\beta-\gamma;\frac{\delta}{\theta},\frac{\delta}{\theta}\Big)\Bigg\} \\
& -(\theta-\delta)^{2\beta}\Bigg\}
\end{aligned} \tag{2.2}$$

where F_δ is a hypergeometric function of several variables.

Proof. For some purposes it may be convenient to use Whittaker's notation. For $\delta>\theta$, UMVUE U_m in (2.1) can be expressed by Whittaker function as follows;

$$\begin{aligned}
U_m = & \frac{\Gamma(n-1)}{\Gamma(n-\beta-\gamma-1)}\delta^{(n-\beta-\gamma)/2}B^\alpha(A/n)^{(n+\beta+\gamma)/2}e^{\delta n/2A} \\
& \times\Big\{(\delta n/A)^{\frac{1}{2}}M_{\lambda_1,\mu_1-\frac{1}{2}}(\delta n/A)-\frac{\alpha}{(n-\beta-\gamma-1)A}M_{\lambda_2,\mu_2-\frac{1}{2}}(\delta n/A)\Big\}
\end{aligned}$$

where $\lambda_1=\frac{(n+\beta-\gamma-1)}{2}, \lambda_2=\frac{(n+\beta-\gamma)}{2}, \mu_1=\frac{(n-\beta-\gamma-1)}{2}$, and $\mu_2=\frac{(n-\beta-\gamma)}{2}$.

We must calculate the following expectation;

$$E(U_m^2) = \frac{\{\Gamma(n-1)\}^2}{\{\Gamma(n-\beta-\gamma-1)\}^2}\delta^{-(n-\beta-\gamma)}E(B^{2\alpha})$$
$$\times\Bigg\{\delta E\Big\{(n/A)^{-(n+\beta+\gamma-1)}e^{\delta n/A}M^2_{\lambda_1,\mu_1-\frac{1}{2}}(\delta n/A)\Big\}$$
$$+\frac{\alpha^2}{n^2(n-\beta-\gamma-1)^2}E\Big\{(n/A)^{-(n+\beta+\gamma-2)}e^{\delta n/A}$$
$$M^2_{\lambda_2,\mu_2-\frac{1}{2}}(\delta n/A)\Big\} - \frac{2\alpha\delta^{\frac{1}{2}}}{n(n-\beta-\gamma-1)}E\Big\{(n/A)^{-(n+\beta+\gamma-3/2)}$$
$$e^{\delta n/A}M_{\lambda_1,\mu_1-\frac{1}{2}}(\delta n/A)M_{\lambda_2,\mu_2-\frac{1}{2}}(\delta n/A)\Big\}\Bigg\}\,.$$

We start with the formula 7.622.3 in Gradshteyn and Ryzhik(1965);

$$\int_0^\infty x^{v-1}e^{-bx}M_{\lambda_1,\mu_1-\frac{1}{2}}(a_1x)\cdots M_{\lambda_n,\mu_n-\frac{1}{2}}(a_nx)dx$$
$$=a_1^{\mu_1}\cdots a_n^{\mu_n}(b+A)^{-v-M}\Gamma(v+M)$$
$$F_A\Big(v+M;\mu_1-\lambda_1,\ldots,\mu_n-\lambda_n;2\mu_1,\ldots,2\mu_n;$$
$$\frac{a_1}{b+A},\ldots,\frac{a_n}{b+A}\Big),\quad Re(v+M)>0,$$
$$Re\Big(b\pm\frac{1}{2}a_1\pm\cdots\pm\frac{1}{2}a_n\Big)>0,$$

where $M=\mu_1+\mu_2+\cdots+\mu_n, A=\frac{1}{2}(a_1+\cdots+a_n)$.
By the proposition, $E(B^{2\alpha})=(\theta n\lambda^{2\alpha})/(\theta n-2\alpha)$ for $\theta n>2\alpha$ and hence

$$E(U_m^2)=\frac{n\Gamma(n-1)\Gamma(n-2\beta-2\gamma-1)\lambda^{2\alpha}\theta^{2\beta+2\gamma+1}}{\{\Gamma(n-\beta-\gamma-1)\}^2(\theta n-2\alpha)}$$
$$\Bigg\{F_\delta\Big(n-2\beta-2\gamma-1;-\beta,-\beta;n-\beta-\gamma-1,n-\beta-\gamma-1;\frac{\delta}{\theta},\frac{\delta}{\theta}\Big)$$
$$+\frac{\alpha^2(n-2\beta-2\gamma)(n-2\beta-2\gamma-1)}{n^2(n-\beta-\gamma-1)^2\theta^2}F_\delta\Big(n-2\beta-2\gamma+1;-\beta,-\beta;$$
$$n-\beta-\gamma,n-\beta-\gamma;\frac{\delta}{\theta},\frac{\delta}{\theta}\Big)-\frac{2\alpha(n-2\beta-2\gamma-1)}{n(n-\beta-\gamma-1)\theta}$$
$$F_\delta\Big(n-2\beta-2\gamma;-\beta,-\beta;n-\beta-\gamma-1,n-\beta-\gamma;\frac{\delta}{\theta},\frac{\delta}{\theta}\Big)\Bigg\}\,.$$

Theorem 3. For any real numbers α, β, γ, and $\delta < 0, n \geq 2$ such that $n-2\beta-2\gamma-1>0$, $n-\beta-\gamma-1>0$, $\theta n > 2\alpha, and\ \theta \geq 2\delta$, the variance of the UMVUE U_m in (2.1) is given by

$$
\begin{aligned}
Var(U_m) = \lambda^{2\alpha}\theta^{2\gamma}\Bigg\{ & \frac{n\Gamma(n-1)\theta^{n-2\gamma}\Gamma(n-2\beta-2\gamma-1)}{\{\Gamma(n-\beta-\gamma-1)\}^2(\theta n-2\alpha)(\theta-2\delta)^{n-2\beta-2\gamma-1}} \\
& \Bigg\{F_{-\delta}\Big(n-2\beta-2\gamma-1; n-\gamma-1, n-\gamma-1; n-\beta-\gamma-1, \\
& n-\beta-\gamma-1; \frac{-\delta}{\theta-2\delta}, \frac{-\delta}{\theta-2\delta}\Big) + \frac{\alpha^2(n-2\beta-2\gamma)(n-2\beta-2\gamma-1)}{n^2(n-\beta-\gamma-1)^2(\theta-2\delta)^2} \\
& F_{-\delta}\Big(n-2\beta-2\gamma+1; n-\gamma, n-\gamma; n-\beta-\gamma, n-\beta-\gamma; \\
& \frac{-\delta}{\theta-2\delta}, \frac{-\delta}{\theta-2\delta}\Big) - \frac{2\alpha(n-2\beta-2\gamma-1)}{n(n-\beta-\gamma-1)(\theta-2\delta)} F_{-\delta}\Big(n-2\beta-2\gamma; \\
& n-\gamma-1, n-\gamma; n-\beta-\gamma-1, n-\beta-\gamma; \frac{-\delta}{\theta-2\delta}, \frac{-\delta}{\theta-2\delta}\Big)\Bigg\} \\
& -(\theta-\delta)^{2\beta}\Bigg\}.
\end{aligned} \tag{2.3}
$$

Proof. For $\delta < 0$, from the well-known relation between ${}_1F_1$ and $M_{\lambda,\mu}(.)$, the UMVUE U_m in (2.1) can be expressed by Whittaker function as follows;

$$
\begin{aligned}
U_m = & \frac{\Gamma(n-1)}{\Gamma(n-\beta-\gamma-1)}(-\delta)^{-(n-\beta-\gamma)/2} B^{\alpha}(A/n)^{(n+\beta+\gamma)/2} e^{\delta n/2A} \\
& \times \Bigg\{(-\delta n/A)^{\frac{1}{2}} M_{-\lambda_1,\mu_1-\frac{1}{2}}(-\delta n/A) \\
& - \frac{\alpha}{(n-\beta-\gamma-1)A} M_{-\lambda_2,\mu_2-\frac{1}{2}}(-\delta n/A)\Bigg\}.
\end{aligned}
$$

From $T = 2\theta n/A \sim \chi^2_{2n-2}$, and formula 7.622.3 in Gradshteyn and Ryzhik

(1965), we obtain the second moment of U_m

$$E(U_m^2) = \frac{\Gamma(n-1)n\theta^n\lambda^{2\alpha}(n-2\beta-2\gamma-1)}{\{\Gamma(n-\beta-\gamma-1)\}^2(\theta n-2\alpha)(\theta-2\delta)^{n-2\beta-2\gamma-1}}$$

$$\Big\{F_{-\delta}\Big(n-2\beta-2\gamma-1; n-\gamma-1, n-\gamma-1; n-\beta-\gamma-1,$$

$$n-\beta-\gamma-1; \frac{-\delta}{(\theta-2\delta)}, \frac{-\delta}{(\theta-2\delta)}\Big) + \frac{\alpha^2(n-2\beta-2\gamma)(n-2\beta-2\gamma-1)}{n^2(n-\beta-\gamma-1)^2(\theta-2\delta)^2}$$

$$F_{-\delta}\Big(n-2\beta-2\gamma+1; n-\gamma, n-\gamma; n-\beta-\gamma, n-\beta-\gamma;$$

$$\frac{-\delta}{(\theta-2\delta)}, \frac{-\delta}{(\theta-2\delta)}\Big) - \frac{2\alpha(n-2\beta-2\gamma-1)}{n(n-\beta-\gamma-1)(\theta-2\delta)} F_{-\delta}\Big(n-2\beta-2\gamma;$$

$$n-\gamma-1, n-\gamma; n-\beta-\gamma-1, n-\beta-\gamma; \frac{-\delta}{(\theta-2\delta)}, \frac{\delta}{(\theta-2\delta)}\Big)\Big\}.$$

From the theorem 1, 2, and 3, we obtained the UMVUE's and their variances for several functions of the parameters θ and λ in the Pareto distribution.

Corollary. UMVUE's for several functions of the parameters θ and λ in the Pareto distribution (1.1) and the variances of these UMVUE's are given by the following;

(1) The UMVUE of the shape parameter, θ, and its variance are

$$(n-2)A/n \quad \text{and} \quad \theta^2/(n-3), \quad n>3.$$

(2) The UMVUE of the scale parameter, λ, and its variance are

$$B\Big(1-\frac{1}{(n-1)A}\Big) \quad \text{and} \quad \frac{\lambda^2}{\theta(n-1)(\theta n-2)}, \quad n>2.$$

(3) The UMVUE of the reciprocal of the shape parameter, θ^{-1}, and its variance are

$$\frac{n}{(n-1)A} \quad \text{and} \quad \frac{1}{(n-1)\theta^2}, \quad n>2.$$

(4) The UMVUE of the reciprocal of the scale parameter, λ^{-1}, and its variance are

$$\frac{1}{B}\Big(1+\frac{1}{(n-1)A}\Big) \quad \text{and} \quad \frac{1}{(n-1)(\theta n+2)\theta\lambda^2}, \quad n>2.$$

(5) The UMVUE of $(1/\theta)^r$ and its variance are

$$\frac{\Gamma(n-1)}{\Gamma(n+r-1)}(n/A)^r \quad \text{and} \quad \frac{1}{\theta^{2r}}\left\{\frac{\Gamma(n-1)\Gamma(n+2r-1)}{\{\Gamma(n+r-1)\}^2}-1\right\}, \quad n>2.$$

(6) The UMVUE of the coefficient of variation(CV) of the Pareto distribution, $\theta^{-\frac{1}{2}}(\theta-2)^{-\frac{1}{2}}$, and its variance are

$$\frac{n}{(n-1)A}{}_1F_1\left(\frac{1}{2};n;2n/A\right)$$

and

$$\frac{n}{(n-1)\theta^2}F_2\left(n+1;\frac{1}{2},\frac{1}{2};n,n;\frac{2}{\theta},\frac{2}{\theta}\right)-\frac{1}{\theta(\theta-2)}, \quad n>2.$$

(7) The UMVUE of CV of the Power-function distribution, $\theta^{-\frac{1}{2}}(\theta+2)^{-\frac{1}{2}}$, and its variance are

$$\frac{n}{(n-1)A}{}_1F_1\left(\frac{1}{2};n;-2n/A\right)$$

and

$$\frac{1}{\theta}\left\{\frac{n\theta^n}{n-1}(\theta+4)^{-(n+1)}F_2\left(n+1;n-\frac{1}{2},n-\frac{1}{2};n,n;\right.\right.$$
$$\left.\left.\frac{2}{\theta+4},\frac{2}{\theta+4}\right)-\frac{1}{\theta+2}\right\}, \quad n>2.$$

(8) The UMVUE of θ/λ and its variance are

$$\frac{(n-2)A+1}{nB} \quad \text{and} \quad \frac{\theta}{\lambda^2}\left\{\frac{n^2\theta^2+n-3}{n(n-3)(\theta n+2)}\right\}, \quad n>3.$$

(9) The UMVUE of the rth moment of the Pareto distribution, μ'_r, and its variance are

$$B^r\left\{{}_1F_1(1;n-1;rn/A)-\frac{r}{(n-1)A}{}_1F_1(1;n;rn/A)\right\}$$

and

$$\lambda^{2r}\theta^2\left\{\frac{n}{(\theta n-2r)\theta}\left\{F_r\left(n-1;1,1;n-1,n-1;\frac{r}{\theta},\frac{r}{\theta}\right)\right.\right.$$
$$+\frac{r^2}{n(n-1)\theta^2}F_r\left(n+1;1,1;n,n;\frac{r}{\theta},\frac{r}{\theta}\right)$$
$$\left.\left.-\frac{2r}{\theta n}F_r\left(n;1,1;n-1,n;\frac{r}{\theta},\frac{r}{\theta}\right)\right\}-(\theta-r)^{-2}\right\}, \quad n>2.$$

(10) The UMVUE of the rth moment of the Power-function distribution, μ'_{-r}, and its variance are

$$B^{-r}\left\{{}_1F_1(1;n-1;-rn/A)+\frac{r}{(n-1)A}{}_1F_1(1;n;-rn/A)\right\}$$

and

$$\frac{\theta^2}{\lambda^{2r}}\left\{\frac{n\theta^{n-2}}{(\theta n+2r)(\theta+2r)^{n-1}}\left\{F_r\left(n-1;n-2,n-2;n-1,n-1;\frac{r}{\theta+2r},\frac{r}{\theta+2r}\right)+\frac{r^2}{n(n-1)(\theta+2r)^2}F_r\left(n+1;n-1,n-1;n,n;\frac{r}{\theta+2r},\frac{r}{\theta+2r}\right)+\frac{2r}{n(\theta+2r)}F_r\left(n;n-2,n-1;n-1,n;\frac{r}{\theta+2r},\frac{r}{\theta+2r}\right)\right\}-(\theta+r)^{-2}\right\},\quad n>2\,.$$

3. Discussion

If X has the probability density function(1.1), then logX has a two-parameter exponential distribution with the location parameter $\log\lambda$ and the scale parameter $1/\theta$. The distribution of 1/X is called the Power-function distribution, this is a special Pearson Type I distribution, and in the case $\theta = 1$ this reduces to the rectangular distribution. The UMVUE's of the rth moment and CV of X and 1/X are given in section 2. For the harmonic mean of X, the UMVUE is obtained by U_m as special case.

We can easily construct in the Power-function distribution the UMVUE's corresponding to those in the Pareto distribution by using the reciprocal relation between these distribution, and we also can easily obtain the UMVUE of the scale parameter, the mean deviation and the rth cumulant($r > 1$) in the two-parameter exponential distribution by U_m and the relation between the Pareto distribution and exponential distribution, for instance, the UMVUE's of the scale parameter and the mean deviation in the exponential distribution are obtained by the UMVUE of the reciprocal of the shape parameter in the Pareto distribution, and the UMVUE of the location parameter in the two-parameter Power-function distribution is obtained by the UMVUE of the reciprocal of the scale parameter.

References

1. Baxter, M.A.(1980). Minimum variance unbiased estimation of the parameters of the Pareto distribution, *Metrika,* 27, 133-138.
2. Epstein, B., and Sobel, M.(1954). Some theorems relevant to life testing from an exponential distribution, *Annals of Mathematical statistics,* 25, 373-381.
3. Gradshteyn, I.S., and Ryzhik, I.M.(1965). *Tables of Integrals Series and Products,* New York: Academic Press.
4. Johnson, N.L., and Kotz, S.(1970). *Continuous Univariate Distribution-1,* New York: John Wiley.
5. Likeš, J.(1969). Minimum variance unbiased estimates of the parameters of Power-function and Pareto's distribution, *Statistische Hefte,* 10, 104-110.
6. Malik, H.J.(1970). Estimation of the parameters of the Pareto distribution, *Metrika,* 15, 126-132.
7. Oberhettinger, F., and Badii, L.(1970). *Tables of Laplace Transforms,* New York: Springer-Verlag.
8. Saksena, S.K., and Johnson, A.M.(1984). Best unbiased estimators for the parameters of a two-parameter Pareto distribution, *Metrika,* 31, 77-83.