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Partially BTIB Designs for Comparing Treatments with a Control†

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ABSTRACT

Bechhofer and Tamhane(1981) developed a theory of optimal incomplete block designs for comparing several treatments with a control. This class of designs is appropriate for comparing simultaneously $p \geq 2$ test treatments with a control treatment (the so-called multiple comparisons with a control (MCC) problem) when the observations are taken in incomplete blocks of common size $K < p+1$.

In this paper we want to extend to partially BTIB designs with two associate classes for the MCC problem. We propose a new class of incomplete block designs that are partially balanced with respect to test treatments. Because the class of designs that we consider is larger than the class of designs in Bechhofer and Tamhane and provides us with designs that improve on the optimal designs in their class. We shall use the abbreviation PBTIB to refer to such designs. We study their structure and give some methods of construction.

Also we describe a procedure for making exact joint confidence statements for the MCC problem in PBTIB Designs with two associate classes. We study Optimality, Admissibility considerations in PBTIB designs with two associate classes.

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1. Introduction

This article deals with the problem of comparing simultaneously several treatments, called test treatments, with a special treatment called the control. For this problem the earliest work was carried out by Dunnett(1955,1964). Dunnett(1955) posed (but did not solve) the problem of the optimal allocations. This optimal allocation problem was solved by Bechhofer and his coworkers(1969,1970,1971). Bechhofer and Tamhane(1981) proposed a new class of incomplete block designs which they referred to a Balanced Treatment Incomplete Block(BTIB) designs for the MCC problem. In the present article, we want to extend to Partially Balanced Treatment Incomplete Block(PBTIB) designs with two associate classes for the MCC problem: we suggest a design that allocates the control treatment occurred with any test treatment an equal number of times in blocks, the test treatments forming a partially balanced incomplete block(PBIB) design with two associate classes in the remaining plots of the blocks. Bechhofer-Tamhane article is basic to the development in the present article. Our goal of this of this paper is the consider a general class of optimal PBTIB designs that is appropriate for the MCC problem. This paper consists of 5 Sections. In Section 2,3,4 and 5, we give the basis theory underlying PBTIB designs, and include some methods of construction. The method of analysis is described. Optimality, Admissibility considerations are discussed in some detail. Examples of article are considered of Group Divisible PBTIB.

2. PBTIB Designs with Two Associate Classes

2.1 PBTIB designs

Let the $p+1$ treatments be indexed $0, 1, \cdots, p$ with 0 denoting the control treatment and $1, \cdots, p$ denoting the $p \geq 2$ tests treatments. Suppose that the experimental units are grouped into b sets, called blocks, of k units each, where

$$k < p + 1 \cdot \tag{2.1}$$

This mean that we are in an incomplete block design situation. $N = kb$ is the total number of experimental units. If Y_{ijh} is response obtained by applying treatment i to the hth plot of block j, then the usual additive linear model (no treatment $\times$ block interaction) of response the we shall deal with is

$$Y_{ijh} = \mu + \tau_i + \beta_j + \epsilon_{ijh} \tag{2.2}$$

with

$$\sum_{i=0}^{p} \tau_i = \sum_{j=1}^{b} \beta_j = 0; i = 0, \cdots, p; j = 1, \cdots, b; h = 1, \cdots, r_{ij}(r_{ij} = 0, 1, \cdots, k-1);$$

where r_{ij} denotes the number of experimental units in block j assigned to treatment i. There is no observation Y_{ijh} if $r_{ij} = 0$. In (2.2), μ denotes the general mean, τ_i the effect of treatment i, β_j the effect of block j, and the ϵ_{ijh} are assumed to iid $N(0, \sigma^2)$ random variables.

It is desired to allocate the treatment $0, 1, \cdots, p$ to the blocks in a way that allows the best possible inference (exact joint confidence statement) on the vector of control-test treatment contrast $(\tau_0 - \tau_1, \cdots, \tau_0 - \tau_p)$ based on their BLUE'S $\hat{\tau}_0 - \hat{\tau}_i (1 \le i \le p)$ in the sence of optimality.

Since it is desired to make a confidence statment (employing one-sided or two-sided interval) that applies simultaneously to all of the p differences $\tau_0 - \tau_i (1 \le i \le p)$, we shall regard our problem as being symmetric in these differences. In the sequel,we consider a class of designs for which

$$Var\{\hat{\tau}_0 - \hat{\tau}_i\} = c^2\sigma^2(\text{say})\ (1 \le i \le p), \tag{2.3a}$$

$$corr\{\hat{\tau}_0 - \hat{\tau}_{i1}, \hat{\tau}_0 - \hat{\tau}_{i2}\} = \rho_1$$

(if the i_1th and i_2th treatments are first associates, $i_1 \neq i_2; 1 \le i_1, i_2 \le p$) (2.3$b$)

and

$$corr\{\hat{\tau}_0 - \hat{\tau}_{i1}, \hat{\tau}_0 - \hat{\tau}_{i2}\} = \rho_2$$

(if the i_1th and i_2th treatments are second associates,$i_1 \neq i_2; 1 \le i_1, i_2 \le p$);

the parameters C^2, ρ_1 and ρ_2 depend on the design employed. We shall refer to such designs as partially Balanced Treatment Incomplete Block(PBTIB) designs with two associate classes since they are partially balanced with respect to (wrt) the test treatments.

The following theorem states the necessary and sufficient conditions that a design must satisfy in order to be a PBTIB design.

Theorem 2.1 : For given (p, k, b) consider a design with the incidence matrix $\{r_{ij}\}$ and suppose the p test treatments constitue a partially balanced association schems with two classes. Let $\lambda_{ii'} = \sum_{j=1}^{b} r_{ij} r_{i'j}$ denote the total number of times that the ith treatment appears with the i'th treatment in the same block the whole design $(i \neq i'; 0 \leq i,\ i' \leq p)$.

Then a design with $\{r_{ij}\}$ is PBTIB if and only if (iff)

$$\lambda_{01} = \cdots = \lambda_{0p} = \lambda_0 \text{ (say)}, \tag{2.4}$$

$\lambda_{ii'} = \lambda_1($ if the ith and i' th treatments are first associates, $i \neq i'$; $1 \leq i, i' \leq p)$ and $\lambda_{ii'} = \lambda_2$ (if the ith and i'th treatments are second associates, $i \neq i'$; $1 \leq i$, $i' \leq p)$.

proof : See Kim(1987) .

Remark 2.1 : We noted that Theorem 2.1 places no restrication on $r_i = \sum_{j=1}^{b} r_{ij} (1 \leq i \leq p)$, the number of replications of the ith test treatment, and hence a design can be PBTIB without the $r_i (1 \leq i \leq p)$ being equal. Such a design for $(p, k, b) = (4, 3, 8)$ and $\lambda_0 = 2, \lambda_1 = 0, \lambda_2 = 3$ with $r_1 = r_2 = 5, r_3 = 4, r_4 = 6$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 1 & 4 & 2 & 4 \\ 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 \end{bmatrix}.$$

2.2 Generator Designs

For constructing an implementable PBTIB design with two associate classes we begin with the concept of a generator design.

Definition 2.1 : For given (p, k) a generator design is a PBTIB design (not necessarily connected or implementable) such that no proper subset of its blocks forms a PBTIB design under the same association scheme of same order and none of its blocks contains only one of the $p+1$ treatments where we denote that a PTIB design $\left\{ \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 1 & 2 & \cdots & p \end{matrix} \right\}$ is an elementary generator design.

We consider the following example for $k = 2$ and $p \geq 4$ (except prime number)

I. $p = 4, k = 2$ (2.5*a*)

$$D_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 4 & 1 & 2 & 3 \end{bmatrix}, \quad D_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 & 3 & 4 \end{bmatrix}$$

are BTIB and PBTIB designs with $(\lambda_0, \lambda_1, \lambda_2)$=(1,0,0), (0,0,1), (1,1,0), (1,0,1), respectively; however, only D_0, D_1, and D_2 are generator designs.

II. $p = 6, k = 2$ (2.5*b*)

$$D_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 1 & 2 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 & 5 \\ 2 & 3 & 4 & 3 & 5 & 6 & 5 & 6 & 6 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 6 & 4 & 5 & 3 & 1 & 2 \end{bmatrix},$$

$$D_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 2 & 3 & 4 & 3 & 5 & 6 & 5 & 6 & 6 \end{bmatrix}$$

are BTIB and PBTIB designs with $(\lambda_0, \lambda_1, \lambda_2)$ = (1,0,0), (0,0,1), (0,1,0), (1,0,1), (1,1,0), respectively; however, only D_0, D_1, and D_2 are generator designs.

As shown before, for $p \geq 4$ (except prime number), $k = 2$ there are exactly three generator designs. Although, for each $p \geq 4$ (except prime), $k = 2$ there are three generator designs, it is not clear whether for any (p, k) there are only finitely many generator designs.

2.3 Construction of PBTIB designs

Now we consider only implementable PBTIB design. Suppose that for given (p, k) there are n generator designs $D_i (1 \leq i \leq n)$ with two associate classes under the same association scheme of same order.

Let $\lambda_0^{(i)}, \lambda_1^{(i)}$, and $\lambda_2^{(i)}$ be the design prameters associted with D_i,and let b_i be the number of blocks required by $D_i (1 \leq i \leq n)$. Then an implementable PBTIB design $D = U_{i=1}^{n} f_i D_i$ obtained by taking unions of $f_i \geq 0$ replications of $D_i(1 \leq i \leq n)$, at least one of which has $\lambda_0 > 0$, has the design parameters $\lambda_0 = \sum_{i=1}^{n} f_i\lambda_0^{(i)}, \lambda_1 = \sum_{i=1}^{n} f_i\lambda_1^{(i)}, \lambda_2 = \sum_{i=1}^{n} f_i\lambda_2^{(i)}$ and requires $b = \sum_{i=1}^{n} f_i b_i$ blocks. The set D_i with $f_i > 0$ will be refered to as the support of D. For instance, from D_0, D_1 of (3.4), implementable PBTIB designs of the type $D = f_0 D_0 U f_1 D_1$ can be constructed for $f_0 \geq 1, f_1 \geq 0$; the corresponding design parameters for $D(p = 4)$ are $\lambda_0 = f_0, \lambda_1 = f_1, b = f_0 p + f_1 p$. To this end, for $k \geq 3$ we now state several methods of constructing generator designs for obtaining implementable PBTIB designs with two associate classes.

Method I : The preceding example suggests the following method of constructing a class of generator designs: For given (p, k), a generator design D_m will have $m+1$ plots in each block assigned to the control treatment; the p test treatments are assigned to the remaining $k-m-1$ plots of the b_m blocks $(0 \leq m \leq k-2)$ in such a way as to form a PBIB design. The generator design D_{k-1} contains no control treatments. For $(p,k) = (4,3)$ we have the following three generator design in this class.

$$\mathrm{D}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix}, \quad \mathrm{D}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad \mathrm{D}_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{bmatrix}. \qquad (2.6)$$

Method II : Consider a group-divisible partially balanced incomplete block (GD-PBIB) design with two associate classes between t treatments in blocks of size k. The association scheme of such a GD-PBIB design can be represented in the form of an $m \times n$ array $(with\ mn = t)$. Any two treatments in the same row of the array are first associates, and those in different rows are second associates.

Case 1. $n = 2$: Suppose that $m \geq k$ and m is positive integer (≥ 3); one can then take $p = 2(m-1)$ and arrange the treatments in order of groups so that the first n treatments form the second group, and so on. We relabel the entires in the remaining groups except the treatments 1 through p by zeros, thus obtaining a PBTIB design. Such a design may not be a generator design and may contain some blocka that must be deleted. After deleting such blocks, a PBTIB design is obtained. By identifying the support of this resulting design, the desired generator design(s) are obtained. Some (or all) of these can usually be obtained by the Method I. To see the use of this case, (1) Consider the GD-PBIB design (for $k = 2, t = 6, m = 3, n = 2, b = 18, \lambda_1 = 2, \lambda_2 = 1$) in the monograph by Caltworthy(1956), which has the following association scheme:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

By relabeling the treatments 5 through 6 by zeros, one obtains the union of a PBTIB design with a design containing one block with only zeros. After that block has been deleted, the support of the remaining PBTIB design consists of the following:

$$2 \text{ replications of } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad (2.7a)$$

$$2 \text{ replications of } \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad (2.7b)$$

$$1 \text{ replications of } \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 4 & 3 \end{bmatrix}. \quad (2.7c)$$

Thus the designs given by (2.7a) through (2.7c) are generator designs for $p = 4,\ k = 2$: the three designs are obtainable by Method I. (2) Consider the GD-PBIB design (for $k = 3, t = 6, m = 3, n = 2, b = 8, \lambda_1 = 0, \lambda_2 = 2$), which has the following association scheme:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

By relabeling the treatments 5 through 6 by zeros, one obtains the union of a PBTIB design. The support of the PBTIB design consists of the following:

$$2 \text{ replications of } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix}. \quad (2.7d)$$

Thus the design is generator design for $p = 4,\ k = 3$: the design is obtainably by Method I. (3) Consider the GD-PBIB design $\#R_1$(for $k = 3, t = 6, m = 3, n = 2, b = 6, \lambda_1 = 2, \lambda_2 = 1$) in the monograph by Bose, Clatworthy, and Shrikhande(1954), which has the following association scheme:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} .$$

By relabeling the treatments 5 through 6 by zeros, one obtains a PBTIB design. The PBTIB design consists of the following:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 3 & 4 & 2 & 2 & 4 & 4 \end{bmatrix} . \tag{2.7e}$$

Thus the design is generator design for $p = 4, k = 3$; the design is not obtainable by Method I.

Case 2. $m \geq 2, n \geq 3$: If $p \geq 4$(even); one can then take $p = m(n-1)$ (or $n(m-1)$) and arrange the treatments in order of columns so that the first m(or n) treatments from the first column, the next m(or n) treatments from second column, and so on. We relabel the entries in the remaining columns except the treatments 1 through p by zeros, thus obtaining a PBTIB design. Such a design may not be a generator design and may contain some blocks that must be deleted. After deleting such blocks, a PBTIB design is obtained. By identifying the support of this resulting design, the desired generator design(s) are obtained; some (or all) of these can usually be obtained by the method I.

For instance, consider the GD-PBIB design $\#R_{12}$(for $k = 3, t = 9, m = 3, n = 3, b = 27, \lambda_1 = 3, \lambda_2 = 2$) in the monography by Bose, Clatworthy, and Shrikhande(1954), which has the following association scheme:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} .$$

By relabeling the treatments 7 through 9 by zeros, one obtains the union of a PBTIB design with a design containing one block with only zeros. After that block has been deleted, the support of the remaining PBTIB design consists of the following:

$$3 \text{ replications of } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} , \tag{2.8a}$$

$$\text{2 replications of} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 & 2 & 5 \\ 5 & 6 & 4 & 6 & 4 & 5 & 3 & 6 \end{bmatrix}, \qquad (2.8b)$$

Thus the design given by (2.8a) through (2.8b) are generator designs for $p = 6, k = 3$;the first design is obtainable by Method I, while the second design is not.

Method III : Consider a triangular partially balanced incomplete block(TR-PBIB) design with two associate classes between t treatment in blocks of size k. The association scheme of such a TR-PBIB design can be represented in the form of an $n \times n$ symmetric array $(t = n(n-1)/2, n > 4)$ of n rows and n columns by leaving the main diagonal empty, and writing the treatments in the places above the diagonal (repeating them symmetrically below the diagonal). Two treatments are first associates if they appear in the same row (or the same column) of the array; otherwise they are second associates. For $n > 5$ one can take $p = t-$(the elements in the last column (or row) and arrange the treatments in the places above the diagonal in order of columns so that the first one treatment form the first column, the next two treatments form the second column, and so on (repeating them symmetrically below the diagonal). We relabel the entries in the remaining columns except the treatments 1 through p by zeros, thus obtaining a PBTIB design. Such a design may not be a generator design and may contain some blocks that must be deleted. After deleting such blocks, a PBTIB design is obtained. By identifying the support of this resulting design, the desired generator design(s) are obtained; some (or all) of these can usually be obtained by the Method I. For instance, consider the TR-PBIB design (for $k = 2, t = 15, n = 6, b = 60, \lambda_1 = 1, \lambda_2 = 0$) is the monograph by clatworthy(1973), which has the following association scheme:

$$\begin{bmatrix} \star & 1 & 2 & 4 & 7 & 11 \\ 1 & \star & 3 & 5 & 8 & 12 \\ 2 & 3 & \star & 6 & 9 & 13 \\ 4 & 5 & 6 & \star & 10 & 14 \\ 7 & 8 & 9 & 10 & \star & 15 \\ 11 & 12 & 13 & 14 & 15 & \star \end{bmatrix}.$$

By relabeling the treatments 11 through 15 by zeros, one obtains the union of a PBTIB design with a design containing one block with only zeros. After that block has been deleted, the support of the remaining PBTIB consists of the following:

$$2 \text{ replications of } \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{bmatrix}, \tag{2.9a}$$

1 replication of

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 \\ 2 & 3 & 4 & 5 & 7 & 8 & 3 & 4 & 6 & 7 & 9 & 5 & 6 & 7 & 8 & 9 & 10 & 5 & 6 & 6 & 8 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6 & 7 & 7 & 8 & 9 & 10 & 10 \\ 9 & 10 & 8 & 10 & 9 & 7 & 8 & 9 \end{bmatrix}. \tag{2.9b}$$

Thus the designs given by (2.9a) through (2.9b) are generator designs for $p = 10, k = 2$:the two designs are obtainable by Method I.

Method IV : Consider a latin aquare partially balanced incomplete block (LS-PBIB) design with two associate classes between t treatments in blocks of size k. The association scheme of such a LS-PBIB design can be represented in the form of an $n \times n$ array ($t = n^2$). Two treatments are first associates if they appear in the same row or in the same column of the array; otherwise they are second associates. For $n \geq 4$ one can take $p = t-$(the elements in the last column and row) and relabel the entries in the last column and row by zeros. We arrange the treatments in the remaining columns and rows in order of rows, thus obtaining a PBTIB design. Such a design may not be a generator design and contain some blocks that be deleted. After deleting such blocks, a PBTIB design is obtained. By identifying the support of this resulting design, the desired generator design(s) are obtained; some (or all) of these can usually be obtained by the Method I. For instance, consider the LS-PBIB design #LS3 (for $k = 2, t = 16, n = 4, b = 48, \lambda_1 = 1, \lambda_2 = 0$) in the monograph by Clatworthy(1973), which has the following association scheme:

$$\begin{bmatrix} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}.$$

By relabeling the treatments 10 through 16 by zeros, one obtains the union of a PBTIB design with a design containing one block with only zeros. After that block has been deleted, the support of the remaining partially PBTIB design consists of the following:

$$\text{2 replications of} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}, \qquad (2.10a)$$

$$\text{1 replication of} \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 6 & 7 & 7 & 8 \\ 2 & 3 & 4 & 7 & 3 & 5 & 8 & 6 & 9 & 5 & 6 & 7 & 6 & 8 & 9 & 8 & 9 & 9 \end{bmatrix} (2.10b).$$

Thus the two designs given by (2.10a) through (2.10b) are generator designs for $p = 9$;the designs are obtainable by Method I.

Method V : Suppose that for given (p,k) we have a generator design D_i with $\lambda_0 > 0$.Then a new generator design D_2 for the same (p, k) can be obtained by taking a "complement" of D_1 in the following way : Separate the block of D_1 in different sets so that each block in given set has zero assigned in an equal numbr of plots (0 times, 1 time, etc). For example, consider the design (2.7e) the blocks of which can be seperated into three sets a follows:

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 3 & 4 & 2 & 2 & 4 & 4 \end{bmatrix}.$$

For each set of D write its "complementary" set (with zero assigned in the same number of plots) so that the union of that set with its complementary set forms a generator design ; if $r_{ij} = 0$ or 1 $(1 \leq i \leq p)$ then that union is simply a generator design that can be constructed by Method I. These complementary sets in the present example are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{bmatrix};$$

by taking their union we obtain the generator design D_2. The b-values for D_1 and its complement D_2 are not in general equal, although in the precesent example they are.

3. Joint Confidence Statements

3.1 Expressions joint confidence interval estimates

We first record the expression derived the PBTIB(group divisible, triangular, latin square) for the BLUE $\hat{\tau}_0 - \hat{\tau}_i$ of $\tau_0 - \tau_i$ $(1 \leq i \leq p)$ and their variance, two correlations and average correlation. Let T_i denote the sum of all observations obtaind with the ith treatment $(0 \leq i \leq p)$, and let B denote the sum of all observation in the ith block $(0 \leq j \leq b)$.

Define $B_i^* = \sum_{j=1}^{b} r_{ij} B_j$ and let $Q = kT_i - B_i^* (0 \leq i \leq p)$, and let $S_k(Q_i)$ be $Q_{i1} + Q_{i2} + \cdots + Q_{in_k}$ when $i_1, i_2, \cdots, i_{n_k}$ are k th associates of i th treatment $(k = 1, 2)$.

Then

$$\hat{\tau}_0 - \hat{\tau}_i = \frac{1}{\lambda_0(p+1)(\lambda_0 + p\lambda_1)(\lambda_0 + n_1\lambda_1 + (n_2+1)\lambda_2)}$$

$$\times \Big[Q_0 \lambda_1 [(p+1)\lambda_0 + n_1(p+1)\lambda_1\lambda + \{n_2(p+2) + n_1 + 2\}\lambda_2]$$

$$- Q_i \lambda_0 (p+1)(\lambda_0 + p\lambda_1) - S_1(Q_i)\lambda_0(p+1)(\lambda_1 - \lambda_2) \Big]$$

$$(n_1 + n_2 = p - 1, \ 1 \leq i \leq p). \tag{3.1}$$

Also,

$$Var\{\hat{\tau}_0 - \hat{\tau}_i\} = c^2 s^2 \ (1 \leq i \leq p), \tag{3.2}$$

$\rho_1 = corr(\hat{\tau}_0 - \hat{\tau}_{i1}, \hat{\tau}_0 - \hat{\tau}_{i2})$, $(i1 \neq i2 \ ; \ 1 \leq i1, \ i2 \leq p \ ; \ i1, i2 \ ;$ first associates),

and

$\rho_2 = corr(\hat{\tau}_0 - \hat{\tau}_{i1}, \hat{\tau}_0 - \hat{\tau}_{i2})$, $(i1 \neq i2 \ ; \ 1 \leq i1, \ i2 \leq p \ ; \ i1, i2 \ ;$ second associates),

The expressions for (3.1) through (3.2) are derived in the Appendix of Kim(1987).

Table 1. Analysis of Variance Table for Partially BTIB Designs

Source of Variation	Sum of Squares	d.f.
Treatments (Adjusted)	$\sum_{i=1}^{p} \frac{Q_i^2}{kA} + \sum_{i=1}^{p} \frac{Q_i S_1(Q_i)(\lambda_1\lambda_2)}{kA(\lambda_0+p\lambda_1)}$ $- \sum_{i=1}^{p} \frac{(\lambda_1-\lambda_2)Q_0[(\lambda_0-\lambda_2)S_1(Q_i)-n_1(\lambda_0-\lambda_1 Q_i)]}{kA(\lambda_0+p\lambda_1)(p+1)\lambda_0}$ $-\frac{PQ_0^2 D(D+(\lambda_1-\lambda_2)(p+1)\lambda_0)}{kA(\lambda_0+p\lambda_1)^2\lambda_0^2}$ $+\frac{(p\lambda_0+\lambda_2)Q_0^2}{k(p+1)^2\lambda_0^2}$	p
Blocks	$\frac{1}{k}\sum_{b=1}^{j} B_{j=1}^2 - \frac{G^2}{N}$	b-1
Error	(by subtraction)	N-p-b
Total	$H - \frac{G^2}{N}$	N-1

An unbiased estimate S_v^2 of σ^2 based on $v = N-p-b$ degrees of freedom(df) can be computed as $SS_{error}/(n-p-b)$ where SS_{error} can be obtained by subtraction (as in partially BIB designs) from Table1.

A and D denote $(\lambda_0 + n_1\lambda_1 + (n_2+1)\lambda_2)$ and $(\lambda_0-\lambda_2)(\lambda_0-p\lambda_1) - (\lambda_1 - \lambda_2((n_2+2)\lambda_0 + n_1\lambda_1))$, respectively. The expressions in table are derived in the Appendix of this paper.

We note that if $\lambda_0, \lambda_1, \lambda_2$, then $SS_{treat(adj.)}$ reduces to

$$\frac{Q_0^2(p+1)^2\lambda_0\lambda_1}{k(p+1)^2\lambda_0^2(\lambda_0+p\lambda_1)} + \frac{\sum_{i=1}^{p} Q_i^2}{k(\lambda_0+p\lambda_1)}$$

(i.e., the same expressions as for a BTIB design).

Remark 3.1 : For many PBTIB designs (group divisible, triangular, latin square) we have $r_{ij} > 1$, and thus within-block replication occurs. For such designs the sum of squares(SS) for error can be partitioned into SS due to "pule error" and SS due to "interaction", and this decomposition can be used in testing the additivity assumption or the assumption of block-to-block variance homogeneity. Such tests are not pursued in this paper.

Remark 3.2 : It is easily shown for PBTIB designs that the $\hat{\tau}_i$ have variance$=\eta^2\sigma^2$(say), two correlations ν_1, ν_2, and use have the relationship

$$\begin{aligned} Var\{\hat{\tau}_i - \hat{\tau}_j\} &= 2(1-\nu_1)\eta^2\sigma^2 \\ &= 2(1-\rho_1)c^2\sigma^2, \\ &\quad (i \neq j; 1 \leq i,j \leq p;\ \text{first associates}), \end{aligned}$$

and

$$Var\{\hat{\tau}_i - \hat{\tau}_j\} = 2(1-\nu_2)\eta^2\sigma^2$$
$$= 2(1-\rho_2)c^2\sigma^2,$$
$$(i \neq j; 1 \leq i,j \leq p;\ \text{second associates}). \quad (3.3)$$

Thus the relative precision of the estiamtors $\hat{\tau}_0 - \hat{\tau}_i (1 \leq i \leq p)$ for the MNN problem w.r.t the estimators $\hat{\tau}_i - \hat{\tau}_j (i \neq j, 1 \leq i,j \leq p)$ for the pairwise comparisons among the p test treatments is given by

$$\frac{Var\{\hat{\tau}_i - \hat{\tau}_j\}}{Var\{\hat{\tau}_0 - \hat{\tau}_i\}} = 2(1-\rho_1),$$
$$(\text{i, j : first associates})$$

and

$$\frac{Var\{\hat{\tau}_i - \hat{\tau}_j\}}{Var\{\hat{\tau}_0 - \hat{\tau}_i\}} = 2(1-\rho_2),$$
$$(\text{i, j : second associates}). \quad (3.4)$$

Note that the relative precision is $><1$ depending on whether $\rho_t >< \frac{1}{2}$ $(t = 1, 2)$.

4. OPTIMAL DESIGNS

4.1 Optimality

Now we consider a rational for not only choosing a designs from a set of competing PBTIB designs with two associate classes but also comparing PBTIB designs with two associate classes with BTIB designs,

We consider here the case of confidence intervals; σ^2 is assumed to be known.

We limit consideration to confidence intervals of the form $\{\tau_0 - \tau_i \geq \hat{\tau}_0 - \hat{\tau}_i - d\ \ (1 \leq i \leq ip)\}$ or $\{\hat{\tau}_0 - \hat{\tau}_i - d \leq \tau_0 - \tau_i \leq \hat{\tau}_0 - \hat{\tau}_i + d\ \ (1 \leq i \leq p)\}$, where $d > 0$ is a specified "yardstick" associated with the common width of confidence intervals. The probability P associated with this joint confidence statment can be writtern as the following.

(I) One-sided intervals:

$$
\begin{aligned}
P &= Pr\{\tau_0 - \tau_i \geq \hat{\tau}_0 - \hat{\tau}_i \ \ (1 \leq i \leq p)\}\\
&= Pr(z_{ij} \leq \frac{d}{c\sigma},\ i = 1, \cdots, n_1 + 1,\ j = 1, \cdots \frac{n_2}{n_1+1} + 1)\\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{h=1}^{\frac{n_2}{n_1+1}+1} \phi^{n_1+1}\Bigg[\frac{1}{\sqrt{1-\rho_1}}\sqrt{\rho_1 - \rho_2}\\
&\times \left\{\sum_{j=1}^{h-1} x_j \frac{\rho_2}{\sqrt{\rho_1 + (j-2)\rho_2}\sqrt{\rho_1 + (j-1)\rho_2}}\right\} + x_h \frac{\sqrt{\rho_1 + (h-1)\rho_2}}{\sqrt{\rho_1 + (\eta - 2)\rho_2}} + \frac{d}{c\sigma}\Bigg]\\
&\times d\phi(x_1) \cdots, d\phi(x_{(n2/n1+1)+1})
\end{aligned} \tag{4.1}
$$

(II) Two-sided intervals :

$$
\begin{aligned}
P &= Pr\{\hat{\tau}_0 - \hat{\tau}_i - d \leq \hat{\tau}_0 - \hat{\tau}_i + d \ \ (1 \leq i \leq p)\}\\
&= Pr\{-\frac{d}{c\sigma} \leq z_{ij} \leq \frac{d}{c\sigma},\ \ i = 1, \cdots, n_1 + 1, j = 1, \cdots, \frac{n_2}{n_1+1} + 1\}\\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{h=1}^{\frac{n_2}{n_1+1}+1} \Bigg[\phi\Bigg\{\frac{1}{\sqrt{1-\rho_1}}\sqrt{\rho_1 - \rho_2}\\
&\times \left(\sum_{j=1}^{h-1} x_j \frac{\rho_2}{\sqrt{\rho_1 + (j-2)\rho_2}\sqrt{\rho_1 + (j-1)\rho_2}}\right) + x_h \frac{\sqrt{\rho_1 + (h-1)\rho_2}}{\sqrt{\rho_1 + (h-2)\rho_2}} + \frac{d}{c\sigma}\Bigg\}\\
&- \phi\Bigg\{\frac{1}{\sqrt{1-\rho_1}}\sqrt{\rho_1 - \rho_2}\left(\sum_{j=1}^{h-1} x_j \frac{\rho_2}{\sqrt{\rho_1 + (j-2)\rho_2}\sqrt{\rho_1 + (j-1)\rho_2}}\right)\\
&+ x_h \frac{\sqrt{\rho_1 + (h-1)\rho_2}}{\sqrt{\rho_1 + (h-2)\rho_2}} - \frac{d}{c\sigma}\Bigg\}\Bigg]\\
&\times d\phi(x_1), \cdots, d\phi(x_{(n2/n1+1)+1}))
\end{aligned}
$$

where $(Z_1, \cdots, Z_p)$ has a p-variate standard normal distribution with ρ_1, ρ_2, and $\phi(\cdot)$ denotes the standard univariate normal distribution function. Note that for given P and specified d/σ the probability P of (5.1) depends on the PBTIB design employed only through c, ρ.

This fact will facilitate comparisons between PBTIB designs. In these comparisons we mainly restrict consideration to PBTIB designs with possibly unequal b values for given (p,k) ; however, our results can be extended to the comparison of PBTIB designs with unequal k values.

We start by marking the following definition :

Definition 4.1 : For given (p,k) and specified d/σ the PBTIB design that achieves a joint confidence coefficient $P \geq 1-\alpha$ with the smallest b is said to be optimal for that value of $1-\alpha$.

To determine the optimal PBTIB design for given $(p,k), 1-\alpha$ and specified d/σ, one would proceed as follows :

Find the design that for given (p,k,b) and maximized P, and then very b to find the smallest b for which the maximum P is $\geq 1-\alpha$.

In the search for the optimal design for given (p,k), it is desirable to eliminate from consideration certain designs that are uniformly dominated by other designs and hence cannot be optimal for any d/σ or $1-\alpha$.

4.2 Admissible Designs and Minimal Complete Class of Generator Designs

We define the concept of inadmissible and admissible designs. As noted before, these concepts are motivated by problem of joint confidence interval estimation of the $\tau_0 - \tau_i$

Definition 4.2 : Suppose that for given (p,k) we have two PBTIB designs D_1 and D_2 with parameters $(b_1, \lambda_0^{(1)}, \lambda_1^{(1)}, \lambda_2^{(1)}, c_1^2, \rho_1)$ and $(b_2, \lambda_0^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)}, c_2^2, \rho_2)$ and with $b_1 \leq b_2, D_2$ is inadimissable wrt D_1 iff for every d and σ, D_1 yields a confidence coefficient P at least as large as (larger than) that yielded by D_2 when $b_1 < b_2 (b_1 = b_2)$. If a design is not inadmissible, then it is said to be admissible. If $b_1 = b_2,\ c_1^2 = c_2^2,\ \rho_1 = \rho_2$ (or equivalently $b_1 = b_2,\ \lambda_0^{(1)} = \lambda_0^{(2)},\ \lambda_1^{(1)} = \lambda_1^{(2)},\ \lambda_2^{(1)} = \lambda_2^{(2)}$),then D_1 and D_2 are equivalent. For given (p,k) the candidates for an "optimal" design will be all admissible PBTIB designs that can be constructed by forming unions of replications of all known generator designs for that given (p,k).

Theorem 4.1 : For given (p,k) consider two PBTIB designs D_1 and D_2 with parameters (b_1, c_1^2, ρ_1) and (b_2, c_2^2, ρ_2), respectively. Design D_2 is inadmissible wrt design D_1 iff $b_1 \leq b_2,\ c_1^2 \leq c_2^2$ and $\rho_1 \geq \rho_2$ with at least one inequality strict.

Proof of sufficiency: From (4.1) we see that as C decreases for fixed d, σ and ρ (ρ increases for fixed d,σ and C),the confidence coefficient P increases. The monotonicity wrt ρ follows from Slepian's inequality.

Proof of necessity: Suppose that the confidence coefficient associated with D_1 is larger than the confidence coefficient associated with D_2 for every d and ρ. Then $c_1^2 \leq c_2^2 (\rho_1 \geq \rho_2)$ follows from letting $d \uparrow \infty (d \downarrow 0)$.

For an application of this theorem consider the following two GD-PBTIB designs for $(p, k) = (4, 3)$;

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 3 & 4 & 2 & 2 & 4 & 4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 2 & 4 \\ 1 & 2 & 3 & 4 & 3 & 4 & 4 \end{bmatrix}.$$

For these designs $\lambda_0^{(1)} = \lambda_0^{(2)} = 2$, $\lambda_1^{(1)} = \lambda_1^{(2)} = 2$, $\lambda_2^{(1)} = \lambda_2^{(2)} = 1$ and $b_1 = 6 < b_2 = 7$. Hence $c_1^2 = c_2^2 = 11/16$, $\rho_1 = \rho_2 = 13/33$, and thus both D_1 and D_2 yield the same P for every d and σ ;however D_2 is admissible wrt D_1 because D_2 requires a larger total number of observation than does D_1.

Remark 4.1 : For given (p,k) Definitions 4.1 and 4.2 can be considered to the comparison of PBTIB designs not only with unequal b values but also with unequal k values. Such comparisons would be of interest to the experimenter who is faced with the choice of block size,subject to the restriction that the common block size $k < p + 1$. In this case, for given P and specified d/σ the PBTIB design that achieves a joint confidence coefficient $P \geq 1 - \alpha$ with the smallest $N = kb$ is said to be optimal for that value of $1 - \alpha$. If this more general definition of optimality is used, the characterization of inadmissibility given by Theorem 5.1 would be modified as follows: For given P consider two PBTIB designs D_1 and D_2 with parameters $(b_1, k_1, c_1^2, \rho_1)$ and $(b_2, k_2, c_2^2, \rho_2)$, resepctively. Design D_2 is inadmissible wrt D_1 iff $N_1 = k_1 b_1 \leq N_2 = k_2 b_2$, $c_1^2 \leq c_2^2$, $\rho_1 \geq \rho_2$ with at least one inequality strict.

For an application of this more general definition consider the following two GD-PBTIB designs for $(p, k) = (4, 4)$ and $(p, k) = (4, 3)$;

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 3 & 2 & 2 & 3 & 3 \\ 2 & 4 & 3 & 4 & 4 & 4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ 3 & 4 & 2 & 2 & 2 & 3 & 4 & 3 & 4 & 4 & 4 & 4 \end{bmatrix}.$$

Both D_1 and D_2 have $C^2 = 4/13$, $\rho = 19/60$ and therefore achieve the same P. However, D_2 for $k = 3$ has $N = 36$, while D_1 for $k = 4$ has $N = 24$.

Thus although each design is admissible for its own k value, D_2 is inadmissible wrt D_1.

Remark 4.2 : An application of Theorem 5.1 can be extended to the case of joint two-sided confidence intervals. Of course, the optimal designs might be different in the one-sided and two-sided cases for the same (p,k) and d/σ. The same general ideas carry over for unknown σ^2, except that then one would have to specity the expected common "width" of the confidence intervals. We give the definition of the minimal complete class of generator designs.

Definition 4.3 : For given (p,k) the smallest set of generator designs $\{D_i(1 \leq i \leq n)\}$ from which all admissible designs for that (p,k) (except possibly any equivalent ones) can be constructed is called the minimal complete class of generator designs.

We note that for given (p,k), the minimal complete class is unique up to substitution of any generator design in the set by an equivalent one. This fact follows from the definition of the minimal complete class.

To obtain the minimal complete class from a given set of generator designs we proceed in two steps. In the first step we delete any equivalent generator designs (except, of course, one reprensentative of each set of equivalent generator designs). If the union of two or more generator designs yields an equivalent generator design, then we choose to eliminate the latter design from consideration and there by maintain more flexibility for our construction of designs involving larger numbers of blocks. Thus, for example, $p = 4, k = 3$ the designs

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 \\ 3 & 4 & 3 & 4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 4 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 3 & 4 & 2 & 2 & 4 & 4 \end{bmatrix}. \quad (4.2)$$

are all generator designs, but D_3 is equivalent to $D_1 U D_2$: hence we choose to retain only D_1 and D_2 but not D_3. This lead us to the following definition.

Definition 4.4 : If for given (p,k) we have $n \geq 2$ PBTIB generator designs $D_i(1 \leq i \leq n)$, no two of which are equivalent, and no one of which is equivalent to the union of replications of one or more of the other generator designs. Then $\{D_i(1 \le i \le n)\}$ is referred to as the set of nonequivalent generator designs. It would be tempting to eliminate any inadmissible generator designs-room the set of nonequivalent generator designs. However, it is not in general true for given (p,k) that if design D is inadmissible, then every design DUD' is also inadmissible.

Thus, for example, for $p = 4, k = 3$ the GD-PBTIB designs

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 & 2 & 4 & 3 & 4 & 3 & 4 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 2 & 4 \end{bmatrix},$$

which are union of generator designs, have $\lambda_0^{(1)} = 3, \lambda_1^{(1)} = 1, \lambda_2^{(1)} = 2, \lambda_0^{(2)} = 5, \lambda_1^{(2)} = 1, \lambda_2^{(2)} = 0$, and $b_1 = b_2$ Hence $c_1^2 = \frac{16}{33} < c_2^2 = \frac{18}{35}, \rho_1 = \frac{17}{48} > \rho_2 = \frac{1}{18}$, and D_2 is inadmissible wrt D_1. However, $D_2 U D_3$ is admissible wrt $D_1 U D_3$ where

$$D_3 = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 1 & 2 & 4 \\ 3 & 4 & 3 & 4 \end{bmatrix}.$$

Hence in this case it would not be desirable to eliminate D_2 from our set of admissible designs.

In the second step we delete the so-called strongly (S-) inadmissible generator designs from the set of nonequivalent generator designs obtained in the first step. The concept of S-inadmissibility is defined as follows.

Definition 4.5 : If for given (p,k) we have two PBTIB designs D_1 and D_2 (not necessarily generator designs), we say that D_2 is S-inadmissible wrt D_1 if D_2 is admissible wrt D_1, and if for any arbitrary PBTIB design D_3 we have that $D_2 U D_3$ is inadmissible wrt $D_1 U D_3$.

Theorem 4.2 : A sufficient condition for S-inadmissibility of a PBTIB design D_2 wrt a PBTIB design D_1 with the same (p,k) is that

$$b_1 \le b_2,\ \lambda_0^{(1)} = \lambda_0^{(2)},\ \lambda_1^{(1)} \ge \lambda_1^{(2)},\ \lambda_2^{(1)} \ge \lambda_2^{(2)} \tag{4.3}$$

with at least one inequality being strict.

Proof. If $\lambda_1^{(1)} = \lambda_1^{(2)}$, $\lambda_2^{(1)} = \lambda_2^{(2)}$ and $b_1 < b_2$,then the result is obvious. If $\lambda_1^{(1)} > \lambda_1^{(2)}$ and $\lambda_2^{(1)} > \lambda_2^{(2)}$, then the result follows from the fact that for fixed λ_0 the parameter C^2 is a deceasing function of λ_1, λ_2 and ρ is an increasing function of $\lambda_1\lambda_2$. As an illustration of λ_1, λ_2 and ρ is an increasing function teo CD-PBTIB design for $(p, k) = (4, 3)$;

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 3 & 4 & 2 & 2 & 4 & 4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 1 & 2 & 3 & 4 & 2 & 4 \end{bmatrix}.$$

For these designs $\lambda_0^{(1)} = 2$, $\lambda_1^{(1)} = 2$, $\lambda_2^{(1)} = 1$, $b_1 = 6$, and $\lambda_0^{(2)} = 2$, $\lambda_1^{(2)} = 2$, $\lambda_2^{(2)} = 0$, $b_2 = 6$. Hence D_1 is S-inadmissible wrt D_2. We use a special case of (4,3) namely $b_1 \leq b_2$, $\lambda_0^{(1)} = \lambda_0^{(2)}$, $\lambda_1^{(1)} = \lambda_1^{(2)}$, $\lambda_2^{(1)} = \lambda_2^{(2)}$ repeatedly in the sequel to decide whether a given design D_2 is S -inadmissible or equivalent wrt another design D_1.

There are certain PBTIB designs that are S-inadmissible as defined before but that can be deleted without any loss from our set of generator designs.

The identification of such designs requires the concept of combination (c-) inadmissibility, which is more general than S-inadmissibility.

Definition 4.6 : Suppose that for given (p,k) we have $n \geq 2$ generator designs. $\{D_i(1 \leq i \leq n)\}$, which are nonequivalent, and none of which is S-inadmissible. The designs D,D',D" described later are constructed from the designs in the set $\{D_i(1 \leq i \leq n)\}$. consider a PBTIB design D, and an arbitrary PBTIB design D'. If for every D' there exists a PBTIB design D" such that D∪D' is either inadmissible wrt D" or equivalent to D", and D is not included in D", then we say that D is C-inadmissible wrt the set $\{D_i(1 \leq i \leq n)\}$

Remark 4.4 : If a design $\{D_1, \cdots, D_n\}$ that contains only generator designs that are nonequivalent and none of which is S-inadmissible, and if D_i is C-inadmissible wrt that set, then D_i can be deleted from the set, and we shall say that D_i is C-inadmissible wrt the set $\{D_i(j \neq i),\ (1 \leq i \leq n)\}$.

Remark 4.5 : We point out some critical distinctions between S-inadmissible and C-inadmissible designs. First, we note that Theorem 5.2 provides an easy way of checking whether certain PBTIB designs are S-inadmissible wrt certain other PBTIB designs for that (p,k). On the other hand, in order to identify a C-inadmissible design it is necessary to examine every different elementary combination of generator designs, and in some cases higher order combinations, and show that each such combination leads to admissible or equivalent designs. We also note that unions of certain designs with C-inadmissible designs may be admissible, but each such admissible designs may be admissible, design is equivalent to some other design not involing that C-inadmissible design. Such a possibly cannot arise with an S-admissible design. In the equel, we point out that if a design is identified as being S-inadmissible using the sufficient condition of Theorem 4.2, then that design can be permanently deleted without loss, even if it is not known whether the set $\{D_i(1 \leq i \leq n)\}$, contains all generator designs for given (p,k). This is in contrast to the situation concerning a C-inadmissible design, which is defined wrt the set $\{D_i(1 \leq i \leq n)\}$. A design can be C-inadmissible wrt $\{D_i(1 \leq i \leq n)\}$, but not so wrt $\{D_i(1 \leq i \leq n+1)\}$ where there this new set contains then n original designs plus one additional one and consists of *n+1* designs that are nonequivalent, none of which is S-inadmissible.

Thus a C-inadmissible design can be eliminated unless it is known that $\{D_i(1 \leq i \leq n)\}$ contains all nonequivalent and non S-inadmissible generator designs for the particular (p,k) of interest. For $p \geq 4$, and $k = 2$ these sets are given by (2.5a), and (2.5b), respectively.

As ststed before, if the set $\{D_1, \cdots, D_n\}$contains all generator designs for given (p,k), and if $\{D_{i1}, \cdots, D_{im}\}$ with $m \leq n$ is the subset that contains all nonequivalent, none-S-inadmissible and non-C-inadmissible generator designs, then the latter set will referred to as a minimal complete class of generator designs for given (p,k). The designs in the minimal complete class will serve as building blocks for all PBTIB designs which will be of interest to us in our search for the optimal design. we illustrate Definition 4.3 by giving in the following Tables 2.3 and 4 our conjectured minimal complete class of generator designs for $k = 3, p = 4$. Using this tables we have computed catalog of admissible designs for each b and also optimal designs for selected d/σ and $1 - \alpha$ in the Table 5.

Table 2. Conjectured minimal complete class of generator designs for $p=4$, $k=3$

Label								b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$	$\lambda_2^{(i)}$
	0	0	0	0							
D_1	0	0	0	0				4	2	0	0
	1	2	3	4							
	0	0									
D_2	1	2						2	1	1	0
	3	4									
	0	0	1	1	2						
D_3	1	2	2	3	2			5	1	1	2
	2	4	4	4	3						
	0	0	1	1	1	2	3				
D_4	1	3	2	2	1	2	3	7	1	1	3
	2	4	3	4	4	3	4				
	0	0	0	0							
D_5	1	1	2	3				4	2	0	1
	2	4	3	4							
	0	0	0	0	1	2					
D_6	1	1	3	3	1	2		6	2	0	2
	2	2	4	4	4	3					
	1	1	2	3							
D_7	1	1	2	3				4	0	0	2
	2	4	3	4							
	1	1	1	2	1	3					
D_8	2	2	2	3	1	3		6	0	2	3
	3	3	4	4	4	4					
	1	2									
D_9	1	2						2	0	2	0
	3	4									
	1	1	2								
D_{10}	2	3	2					3	0	2	1
	3	4	4								
	1	1	1	2							
D_{11}	2	2	3	3				4	0	2	2
	3	4	4	4							

Table 3. Conjectured minimal complete class of generator designs for p=6, k=3 (m=2, n=3)

Label																b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$	$\lambda_2^{(i)}$
	0	0	0	0	0	0													
D_1	1	1	3	2	2	4										6	2	1	0
	3	5	5	4	6	6													
	0	0	0	0	0	0													
D_2	0	0	0	0	0	0										6	2	0	0
	1	2	3	4	5	6													
	0	0	0	1	2	2	1												
D_3	1	3	5	4	3	4	3									7	1	1	1
	2	4	6	6	6	5	5												
	0	0	0	1	1	1	2	3	3	1	1	2	2	3	4				
D_4	1	3	5	2	2	5	5	4	4	1	1	2	2	3	4	15	1	1	3
	2	4	6	3	4	6	6	5	6	4	6	3	5	6	5				
	0	0	0	1	1	2	2	2	2	1	1	1	1	3	3				
D_5	1	3	5	4	4	3	3	4	4	2	3	2	5	4	5	15	1	2	3
	2	4	6	6	6	6	6	5	5	3	4	5	6	5	6				
	0	0	0	0	0	0	1	1	2	2	3	4							
D_6	1	1	3	3	5	5	1	1	2	2	3	4				12	2	0	2
	2	2	4	4	6	6	4	6	3	5	6	5							
	0	0	0	0	0	0	1	1	1	2	2	2							
D_7	1	1	3	3	5	5	3	4	4	3	3	4				12	2	0	2
	2	2	4	4	6	6	6	5	6	5	6	5							
	0	0	0	0	0	0	0	0	0										
D_8	1	1	1	2	2	3	3	4	5							9	3	0	1
	2	4	6	3	5	4	5	6	6										
	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3				
D_9	1	1	1	2	2	3	3	4	4	5	5	1	2	3		15	4	0	2
	2	2	4	4	3	3	6	6	5	5	6	6	6	5	4				
	1	2																	
D_{10}	3	4														2	0	1	0
	5	6																	
	1	1	1	2	3	4	1	2	3										
D_{11}	2	2	4	3	5	5	6	2	3							9	0	1	2
	3	4	5	6	6	6	6	5	4										

	1	1	1	2	3	3	2	4	5													
D_{12}	1	1	1	2	3	3	2	4	5										9	0	0	2
	2	4	6	3	4	6	5	5	6													
	1	1	1	1	1	2	2	2	3	3												
D_{13}	2	2	3	4	5	3	4	4	4	5									10	0	2	2
	3	5	4	6	6	6	5	6	5	6												
	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	4				
D_{14}	2	2	2	2	3	3	4	4	5	3	3	3	4	5	4	4	5	5	18	0	3	4
	3	4	5	6	4	6	5	6	6	4	5	6	5	6	5	6	6	6				

Table 4. Conjectured minimal complete class of generator designs for $p=6$, $k=3$ (m=3, n=2)

Label													b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$	$\lambda_2^{(i)}$
	0	0	0													
D_1	1	2	3										3	1	1	0
	4	5	6													
	0	0	0	0	0	0	1	4								
D_2	1	1	2	2	3	3	2	5					8	2	0	1
	5	6	4	6	4	5	3	6								
	0	0	0	0	0	0	0	0	0	0	0	0				
D_3	1	1	1	1	2	2	2	3	3	4	4	5	12	4	0	1
	2	3	5	6	3	4	6	4	5	5	6	6				
	0	0	0	0	0	0										
D_4	0	0	0	0	0	0							6	2	0	0
	1	2	3	4	5	6										
	1	1	2	3												
D_5	2	5	4	4									4	0	0	1
	3	6	6	5												
	1	2	3													
D_6	1	2	3										3	0	2	0
	4	5	6													
	1	1	1	2	2	3										
D_7	2	4	3	3	5	4							6	0	2	1
	4	5	6	5	6	6										

Table 5. Optimal Designs (Bechhofer and Tamhanei)

kmn	p	0	1	2	b (d/σ)	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
322	4	1	1	0	2	.2329	.2636	.2960	.3300	.3653	.4016						
322	4	2	2	0	4	.2984	.3470	.3979	.4502	.5030	.5553	.6063	.6551	.7011	.7438	.7827	.8177
						(.1880	.2339	.2631	.3051	.3493	.3952	.4422	.4894	.5363	.5822	.6265	.6687)
322	4	2	2	1	6	.4107	.4682	.5262									
						(.3915	.4582	.5257)									
322	4	4	2	1	8			.5952									
								(.5914)									
322	4	4	2	1	10					.7920	.8465	.8902	.9238	.9487	.9665	.9788	.9870
										(.7907	.8388	.8795	.9139	.9402	.9596	.9735	.9831)
322	4	7	3	2	14			.7508	.8228								
								(.7399	.8029)								
kmn	*p*	0	1	2	*b*	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
322	4	8	4	2	16						.9064	.9425	.9664	.9813	.9900	.9950	.9976
											(.9034	.9387	.9628	.9785	.9881	.9938	.9969)
322	4	10	4	3	20			.7646	.8456	.9050	.9451						
								(.7630	.8358	.8932	.9342)						
322	4	11	5	3	22					.9223	.9574	.9781	.9895	.9953	.9980	.9992	.9997
										(.9211	.9556	.9766	.9885	.9947	.9977	.9990	.9996)
322	4	12	6	3	24										.9988	.9996	
															(.9988	.9996)	
kmn	*p*	0	1	2	*b*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
323	6	2	1	0	6	.1828	.2281	.2785	.3333	.3912	.4511	.5114	.5708	.6280	.6819	.7316	.7766
						(.0815	.1060	.1349	.1685	.2065	.2485	.2940	.3424	.3927	.4442	.4959	.5468)
332	6	2	2	1	10		.4177	.4868									
323	6	3	2	1	13								.8707	.9078	.9362	.9571	.9719
332	6	4	2	1	14						.7862	.8442	.8902	.9252	.9506	.9684	.9804
											(.7751	.8250	.8671	.9016	.9289	.9499	.9656)
323	6	3	3	1	15					.7400	.8025						
										(.7276	.7996)						
332	6	3	3	2	17			.6349	.7122								
								(.6063	.6576)								
332	6	6	2	1	18										.9783	.9878	.9934
															(.9773	.9869	.9927)
332	6	4	4	2	20					.8290	.8826						
332	6	6	4	1	20							.9249	.9544	.9734	.9852	.9921	.9959
332	6	5	3	2	21			.6944	.7791	.8478	.9000	.9374	.9626	.9787	.9884	.9939	.9970
								(.6911	.7640	.8261	.8764	.9153	.9047	.9671	.9812	.9897	.9945)
332	6	4	4	3	24		.6456										
							(.6274)										
kmn	*p*	0	1	2	*b*	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
323	6	3	8	1	25	.3531											
						(.3362)											
323	6	4	5	2	25		.4425										
							(.4424)										
332	6	7	3	2	25						.8293	.8925	.9361	.9642	.9810	.9905	.9955
											(.8276	.8871	.9299	.9587	.9769	.9878	.9938)
323	6	6	4	2	26					.7606	.8396						
										(.7522	.8376)						
332	6	5	5	3	27			.5750									
								(.5586)									
323	6	7	4	3	28			.6075	.7216	.8156	.8860						
								(.5900	.6855	.7696	.8391)						
332	6	9	3	2	29								.9583	.9788	.9899	.9955	.9981
													(.9578	.9781	.9893	.9951	.9979)

323	6	7	5	3	30						.8977	.9424	.9698	.9852	.9932	.9971	.9988
											(.8736	.9276	.9617	.9810	.9911	.9961	.9984)
332	6	5	5	4	31			.6158									
								(.6087)									
323	610	3	2	32						.8926	.9411	.9700	.9858				
										(.8868	.9335	.9635	.9812)				
332	6	7	7	3	33				.7380								
									(.7366)								
kmn	*p*	0	1	2	*b*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
323	6	4	10	2	35	.2270											
						(.2189)											
332	6	10	4	3	36							.8575	.9210	.9587	.9810	.9918	.9967
												(.8567	.9182	.9568	.9790	.9905	.9960)
323	6	9	5	3	37						.7754	.8643					
											(.7696	.8634)					
332	6	6	6	5	38					.6717							
										(.6586)							
323	6	9	6	3	39						.7904						
											(.7872)						
332	6	11	5	3	39							.8778					
												(.8685)					
323	6	12	5	2	40										.9873	.9949	.9981
															(.9873	.9949	.9981)
332	6	7	7	5	41				.5705								
									(.5688)								
332	6	13	9	2	43									.9757	.9899	.9962	
														(.9738	.9890	.9957)	

5. Discussion

We introduced a new general class of incomplete block desugns that are appropriate for use in the MCC problem. We refer to these as partially balanced treatment incomplete block (PBTIB) designs. The basis results concerning the structure of such designs are derived, and the properties of the relevant estimates obtained with such designs are given. Admissibility and inadmissibility of these designs are defined, and these criteria are used to eliminate inferior designs. In the search for optimal designs it suffices to restrict consideration to admissible designs.

It is shown how the concept of S-inadmissibility C-inadmissibility can be used to obtain a minimal complete class of generator designs from which catalogs of admissible designs can be constructed.

The combinatorial problem of constructing all PBTIB designs for given (p,k,b), and the procedure for choosing an optimal design from such a set, are not solved in the present paper. however, some methods of design construction are given.

The aforementioned problems are related in the sense that to solve the optimization part completely one must have constructed most, if not all, generator designs for given (p,k); the problem of determining how many generator designs exist for arbitrary (p,k), and then enumerating them, appears to be a formidable one.

References

1. Bechhofer, R.E.(1981), "Incomplete Block Designs for Comparing Treatments with a Control:General Theory, " *Technometrics 23*, 45-57.
2. Clatworthy, W.H.(1973), "Tables of Two-associate-class Partially Balanced Designs," *Applied Mathematics Series 63*, Nat. Bur. Stand.(U.S.).
3. Kim, K.H.(1986), "On Partially Balanced Treatment Incomplete Block Designs," Ph.D. Dissertation, Korea. Korea Univ., Seoul, Korea.
4. Kwang-Hun Kim and John Stufken(1990), "Group Divisible Treatment Designs for Comparing Treatments with a Control". Submit in Journal.
5. Notz, W.I., and Tamhane, A.C.(1983), "Balanced Treatment Incomplete Block(BTIB) Designs for Comparing treatments with aControl: Minimal Complete sets of Generator Designs for k=3, p=3(1)10," *Comm. Statist.Theor. Meth 12(12)*, 1391-1412.