HONAM MATHEMATICAL JOURNAL
Volume 12, Number 1. July 1990.

Lower Functions for Increasing Levy Processes

Yun Kyong, Kim

Dept. of Mathematics, Dong-shin Engineering College,
Naju 520—180, Korea

1. Introduction

Let $\{X(t)\ ;\ t\geq 0\}$ be a real valued increasing process with stationary independent increments whose charateristic function is given by

$$E\ \exp(iuX(t))=\exp(tg(u))$$

where $g(u)=ibu+\int_0^\infty (e^{iux}-1)\ d\nu(x)$ and ν is a Levy measure on $(0,\infty)$ satisfying

$$\int_0^\infty (x\wedge 1)\ d\nu(x)<\infty$$

The object of this work is to find a positive nondecreasing function $h(t)$ defined on $[0,\infty]$ with $h(0)=0$. $h(\infty)=\infty$ such that for some positive finite constant C,

(1.1) $\qquad \liminf X(t)/h(t)=C\ a.s.$

both as t tends to zero and infinity.

The results relating to our concerns were previously obtained for the case of stable subordinators by Fristedt [3] and for general subordinators by Fristedt and Pruitt [6]. As another results for subordinators. Blumenthal and Getoor [1] defined an index $\delta\in[0,1]$ and proved that if $h(t)=t^{1/\alpha}$, the lim inf in (1.1) as $t\longrightarrow 0$, is zero for $\alpha>\delta$ and infinity for $\alpha<\delta$. Recently, Wee[12] found a sufficient condition for there to exist lower function for suprema of Levy process.

As a problem related to the lim sup for general Levy process $X(t)$, a necessary and sufficient condition for which there exits $h(t)$ satisfying lim sup $|X(t)|/h(x)=C$ $a.s.$ with positive finite constant C was given by Kim and Wee [8] and satisfiable results concerning upper functions for process having only negative jumps was obtained in [9] under some additional assumption.

Analogous results was obtained for sums of i.i.d. random variables by Pruitt [11]. In fact, the present work was motiviated by his paper [11] and we obtain an alternative approach for the results of Fristedt and Pruitt [6]. Some definitions and preliminary lemmas are given in section 2. The main result of behavior for subordinators is obtained in section 3. Throughout this paper, we will use L_2t for $\log|\log t|$.

2. Preliminaries

We assume throughout that $b=0$ and $\int_0^\infty x\, d\nu(x)=\infty$ since the drift term bt plays a dominant role near zero and if $\int_0^\infty x\, d\nu(x)<\infty$ then $X(t)/t\longrightarrow EX(1)$ as $t\longrightarrow\infty$ *a.s.* by SLLN. And as usual, we assume that we are dealing with a version of the process which has almost all sample functions right continuous and having left limits. For $a>0$, we define

$$G(a)=\int_{x>a} d\nu(x)$$
$$H(a)=a^{-1}\int_{0<x\le a} x\, d\nu(x)$$
$$Q(a)=G(a)+H(a)$$

It is easy to verify that Q is positive, continuous, decreasing and zero at infinity. Also

$$aQ(a)=\int_0^a G(x)dx\downarrow 0 \text{ as } a\downarrow 0 \ (\uparrow\infty \text{ as } a\uparrow\infty) \tag{2.1}$$

We assume that $\nu(0,\infty)=\infty$ since then Q is strictly decreasing and $Q(a)\longrightarrow\infty$ as $a\longrightarrow 0$.

Now we define, for $\delta>0$,

$$Q(a_\delta(t))=\delta t^{-1}L_2t \tag{2.2}$$
$$h_\delta(t)=a_\delta(t)L_2t \tag{2.3}$$

Then (2.1) implies that

$$t^{-1}h_\delta(t)\downarrow 0 \text{ as } t\downarrow 0 \ (\uparrow\infty \text{ as } t\uparrow\infty) \tag{2.4}$$

Next we prove a simple lemma which will give us the necessary estimates for the distribution of $X(t)$.

Lemma 2.1. Let $C_0=1-e^{-1}$ and $a>0$. Then

(a) For any $s>0$, $P\{X(t)\leq C_0 taQ(a)-as\}\leq e^{-s}$

(b) $P\{X(t)\leq a\}\geq 1-C_0^{-1}tQ(a)$

(c) If $tQ(a)$ is sufficiently large and $C_1>1$, $C_2>1$, then

$$P\{X(t)\leq C_1 taQ(a)\}\geq \exp(-C_2 tQ(a))$$

Proof. Letting $u=a^{-1}$ we write

$$Ee^{-uX(t)}=\exp\{t\int_{0<x\leq a}(e^{-ux}-1)\,d\nu(x)+t\int_{x>a}(e^{-ux}-1)\,d\nu(x)\}$$

Then we have

$$Ee^{-uX(t)}\leq e^{-C_0 tQ(a)} \tag{2.5}$$

and

$$Ee^{-uX(t)}\geq e^{-tQ(a)} \tag{2.6}$$

The assertion (a) follows from (2.5) and the elementary inequality

$$P(X\leq C)\leq Ee^{-uX}\cdot e^{uC}$$

To prove (b), we note that for $u=a^{-1}$,

$$\begin{aligned}Ee^{-uX(t)}&=\int_{0<x\leq a}e^{-ux}dP(X(t)\leq x)+\int_{x>a}e^{-ux}dP(X(t)\leq x)\\&\leq P(X(t)\leq a)+e^{-1}\{1-P(X(t)\leq a)\}\end{aligned}$$

Therefore, using (2.6), we obtain

$$\begin{aligned}P(X(t)\leq a)&\geq(1-e^{-1})^{-1}(e^{-tQ(a)}-e^{-1})\\&\geq 1-C_0^{-1}tQ(a)\end{aligned}$$

To prove (c), Integrating by parts we obtain

$$\begin{aligned}Ee^{-uX(t)}&=\int_0^\infty ue^{-ux}P(X(t)\leq x)dx\\&\leq P(X(t)\leq\xi)\int_0^\xi ue^{-ux}dx+\int_\xi^\infty ue^{-ux}dx\\&\leq P(X(t)\leq\xi)+e^{-u\xi}\end{aligned}$$

Set $\xi=C_1 taQ(a)$ and $u=a^{-1}$. Then using (2.6), we have

$$P(X(t)\leq C_1taQ(a))\geq e^{-tQ(a)}-e^{-C_1tQ(a)}$$
$$\geq e^{-C_2tQ(a)}$$

We conclude this section with the well-known lemma which will be used in next section. See [10].

Lemma 2.2. (Borel-Cantelli) Let $\{A_n\}$, $\{B_n\}$ be two sequence of events such that the events $\{A_n\}$ are independent and for each n the A_n, B_n are independent Suppose that $\sum P(A_n)=\infty$ and that $P(B_n)\geq c>0$ for all n.

Then

$$P\{A_n\cap B_n\ i.o.\}>0$$

3. Main result

In this section, we prove that for $h_\delta(t)$ defined as in (2.3)

(3.1) $$\liminf X(t)/h_\delta(t)=C\ a.s.$$

both as $t\longrightarrow 0$ and $t\longrightarrow\infty$, where C is a positive constant. The zero-one laws (Blumenthal near zero and Kolmogorov near infinity) guarantee that the lim inf in (3.1) will be constant. Thus it suffices to show that

(3.2) $$\liminf X(t)/h_\delta(t)>0\ a.s.$$

and

(3.3) $$\liminf X(t)/h_\delta(t)<\infty\ a.s.$$

It turns out that (3.2) holds in general, but (3.3) holds under extra condition about the growth rate of Q. To obtain (3.1), we will use the similar argument as in [11]. Now we start with the result of lim sup.

Theorem 3.1. For any $\delta>0$,

$$\limsup X(t)/h_\delta(t)=\infty\ a.s.$$

both as $t\longrightarrow 0$ and as $t\longrightarrow\infty$.

Proof. We consider for small t first. Let $t_k=2^{-k}$. Then by (2.1) and (2.4), for large k

$$\begin{aligned} a_\delta(t_k)\, Q(a_\delta(t_k)) &\leq \int_0^{h_\delta(t_k)} G(x)\ dx \\ &= \sum_{j=k}^{\infty} \int_{h_\delta(t_{j+1})}^{h_\delta(t_j)} G(x)\ dx \\ &\leq \sum_{j=k}^{\infty} h_\delta(t_j)\ G(h_\delta(t_{j+1})) \\ &\leq t_k^{-1} \cdot h_\delta(t_k) \sum_{j=k}^{\infty} t_j G(h_\delta(t_{j+1})) \\ &\leq 4 t_k^{-1} h_\delta(t_k) \int_0^{t_{k+1}} G(h_\delta(t) dt \end{aligned}$$

This implies that for all large k

$$\delta/4 \leq \int_0^{t_{k+1}} G(h_\delta(t))dt$$

and then

$$\int_0^1 G(h_\delta(t))dt = \infty,$$

which gives the proof for $t \longrightarrow 0$. (see [5] p. 307)
For the behavior of large t, it suffices to show that $\sum_n G(h_\delta(n)) = \infty$.
First we prove that

$$\int_0^{\infty} x\ d\nu(x) = \infty \text{ implies } \int_1^{\infty} Q(x)^{-1}\ d\nu(x) = \infty.$$

To see this, suppose that the assertion fails. Then as $a \longrightarrow \infty$,

$$G(a)/Q(a) \leq \int_a^{\infty} Q(x)^{-1}\ d\nu(x) \longrightarrow 0$$

and it follows on integrating by parts in $\int_1^{\infty} Q(x)^{-1}\ d\nu(x)$ that

$$\int_1^{\infty} G(x)\ dQ(x)^{-1} < \infty$$

If we choose $a > 0$ so large that $G(x)/Q(x) \leq \frac{1}{2}$ for $x > a$ then

$$\begin{aligned} \lim_{y\to\infty} yQ(y)/aQ(a) &= \int_a^{\infty} (xQ(x))'/xQ(x)\ dx \\ &\leq 2\int_a^{\infty} (G(x)/x(Q(x))\ (1 - G(x)/Q(x))\ dx \\ &= 2\int_a^{\infty} G(x)\ dQ(x)^{-1} < \infty \end{aligned}$$

since $Q'(x)=x^{-1}(G(x)-Q(x))$. But this implies $\lim_{y\to\infty} yQ(y)=\int_0^\infty G(x)\, dx<\infty$ which contradicts to $\int_0^\infty x\, d\nu(x)=\infty$

Now if we define $b_\delta(t)$ by $Q(b_\delta(t))=\delta t^{-1}$ then by (2.1)

$$a_\delta(t)\leq h_\delta(t)\leq b_\delta(t) \text{ and so } Q(h_\delta(t))\geq\delta t^{-1}$$

for large t. Thus for a fixed large k.

$$\begin{aligned}\int_{x>h_{\delta(k)}} Q(x)^{-1}\, d\nu(x) &\leq\sum_{n=k}^{\infty} Q(h_\delta(n+1))^{-1}\int I(h_\delta(n)<x\leq h_\delta(n+1))\, d\nu(x)\\ &\leq\delta^{-1}\sum_{n=k}^{\infty}(n+1)\ \{G(h_\delta(n))-G(h_\delta(n+1))\}\\ &=\delta^{-1}kG(h_\delta(k))+\delta^{-1}\sum_{n=k}^{\infty}(n-k+1)\ \{G(h_\delta(n))-G(h_\delta(n+1))\}\\ &=\delta^{-1}kG(h_\delta(k))+\delta^{-1}\sum_{n=k}^{\infty} G(h_\delta(n))\end{aligned}$$

which gives the desired result for $t\longrightarrow\infty$.

Theorem 3.2. For any $\delta>C_0^{-1}$,

$$\liminf X(t)/h_\delta(t)>0 \ a.s$$

both as $t\longrightarrow 0$ and as $t\longrightarrow\infty$.

Proof. We consider for $t\longrightarrow 0$. Let $t_k=\rho^{-k}$ with $\rho>1$. Then by Lemma 2.1 (a),

$$\begin{aligned}&P\{X(t_{k+1})\leq C_0 t_{k+1}\ a_\delta(t_k)\ Q(a_\delta(t_k))-(1+\varepsilon)\ a_\delta(t_k)L_2 t_k\}\\ &\quad\leq(k\ \log\ \rho)^{-(1+\varepsilon)}\end{aligned}$$

for any $\varepsilon>0$. We note that

$$\begin{aligned}&C_0 t_{k+1} a_\delta(t_k)\ Q(a_\delta(t_k))-(1+\varepsilon)\ a_\delta(t_k)\ L_2 t_k\\ &=(C_0\delta\rho^{-1}-1-\varepsilon)\ h_\delta(t_k)\end{aligned}$$

Since $\delta>C_0^{-1}$, we can choose $\varepsilon>0$ and $\rho>1$ such that

$$C_0\delta\rho^{-1}>1+\varepsilon$$

Then for $t_{k+1}\leq t<t_k$ and k sufficiently large,

$$X(t)/h_\delta(t)\geq X(t_{k+1})/h_\delta(t_k)\geq C_0\delta\rho^{-1}-1-\varepsilon>0 \ a.s.$$

which completes the proof.

Next Lemma is necessary to prove (3.3), under the extra assumption on the growth rate of Q near zero and infinity.

Lemma 3.3. (a) Suppose that for some $\varepsilon\in(0,1)$, $Q(x)\geq x^{-\varepsilon}$ for x sufficiently small. Let $t_k=\rho^{-k}$ with $\rho>1$ and $M_1=\rho^{8/\varepsilon}$. Then for large j

$$N_j=\text{Card }\{k: 2^j\leq k<2^{j+1} \text{ and } M_1a_\delta(t_{k+1})\leq a_\delta(t_k)\}\leq 2^{j-1}.$$

(b) Suppose that for some $\varepsilon\in(0,1)$, $Q(x)\leq x^{-\varepsilon}$ for x sufficiently large. Let $t_k=\rho^k$ with $\rho>1$ and $M_2=\rho^{4/\varepsilon}$. Then for large j

$$N_j=\text{Card }\{k: 2^j\leq k<2^{j+1} \text{ and } a_\delta(t_{k+1})\geq M_2\ a_\delta(t_k)\}\leq 2^{j-1}.$$

Proof. (a) We note that $a_\delta(t)\geq t^{2/\varepsilon}$ for t sufficiently small.

Let $r_j=t_k$ for $k=2^j$. Then for j sufficiently large,

$$M_1^{-N_j}\geq M_1^{-N_j}\ a_\delta(r_j)\geq a_\delta(r_{j+1})\geq r_{j+1}^{2/\varepsilon}$$

or

$$-N_j\ \log M_1\geq 2\varepsilon^{-1}\log r_{j+1}=-2\varepsilon^{-1}2^{j+1}\log\rho$$

which gives the desired result.

The proof for (b) is similar to (a).

Theorem 3.4. (a) Suppose that for some $\varepsilon\in(0,1)$, $Q(x)\geq x^{-\varepsilon}$ for x sufficiently small. Then for any $\delta>C_0^{-1}$, there exits a positive finite constant C such that

$$\liminf_{t\to 0} X(t)/h_\delta(t)=C\ a.s.$$

(b) Similarly for $t\longrightarrow\infty$ if there is $\varepsilon\in(0,1)$, such that $Q(x)\leq x^{-\varepsilon}$ for x sufficiently large x.

Proof. It suffices to show that (3.3) holds. We take $\rho>\delta\vee 1$ and let $t_k=\rho^{-k}$. Then by Lemma 3.3, for large j, there exist at least 2^{j-1} values of $k\in[2^j,2^{j+1})$ such that

$$(3.4)\qquad M_1a_\delta(t_{k+1})\geq a_\delta(t_k)$$

We form a subsequence $\{s_n\}$ by taking every j-th one of these t_{k+1} for $j=1,2,\cdots$, there will be at least $j^{-1}2^{j-1}$ values of t_{k+1} with $k\in[2^j,2^{j+1})$ in the subsequence for large j. Now let $\xi\in(\delta\rho^{-1},1)$. By Lemma 2.1 (c) with $C_2=\xi^{-1}$, we have

$$(3.5)\qquad P\{X(t)\leq 2ta_\xi(t)\, Q(a_\xi(t))\}\geq|\log t|^{-1}$$

But for t_{k+1} with $k\in[2^j,2^{j+1})$,

$$|\log t_{k+1}|^{-1}=((k+1)\log\rho)^{-1}\geq(\log\rho)^{-1}\,2^{j-1}.$$

Thus if we sum this for $s_n=t_{k+1}$ with $k\in[2^j,2^{j+1})$ in the subsequence, we will have at least $(4\log\rho)^{-1}\,j^{-1}$. This imples $\sum|\log s_n|^{-1}=\infty$.

On the other hand, for those n with $s_n=t_{k+1}$, $k\in[2^j,2^{j+1})$,

$$s_{n+1}\leq s_n\rho^{-j}$$

and so

$$s_{n+1}s_n^{-1}L_2s_n\leq j\rho^{-j}\longrightarrow 0 \text{ as } j\longrightarrow\infty.$$

Therefore by Lemma 2.1 (b), as $n\longrightarrow\infty$

$$P\{X(s_{n+1})\leq a_\xi(s_n)\}\geq 1-C_0^{-1}\,\xi s_{n+1}s_n^{-1}L_2s_n\longrightarrow 1$$

and by (3.5)

$$\begin{aligned}&P\{X(s_n)-X(s_{n+1})\leq 2s_na_\xi(s_n)\, Q(a_\xi(s_n))\}\\ &\geq P\{X(s_n)\leq 2s_na_\xi(s_n)\, Q(a_\xi(s_n))\}\\ &\geq|\log s_n|^{-1}\end{aligned}$$

Hence Lemma 2.2 and zero-one law imply that

$$P\{X(s_n)-X(s_{n+1})\leq 2s_n\, a_\xi(s_n)\, Q(a_\xi(s_n)),\ X(s_{n+1})\leq a_\xi(s_n) \text{ i.o.}\}=1$$

But this means that infinitely often with probability one

$$(3.6)\qquad \begin{aligned}X(s_n)&\leq 2\xi a_\xi(s_n)\, L_2s_n+a_\xi(s_n)\\ &\leq 2a_\xi(s_n)\, L_2s_n\end{aligned}$$

Now for large n

$$\begin{aligned}Q(a_\delta(\rho s_n))&=\delta\rho^{-1}s_n^{-1}L_2(\rho s_n)\\ &<\xi s_n^{-1}L_2s_n=Q(a_\xi(s_n))\end{aligned}$$

and then by (3.4)

$$a_\xi(s_n)\leq a_\delta(\rho s_n)\leq M_1a_\delta(s_n)$$

This estimate and (3.6) imply that i.o. with probability one

$$X(s_n)\leq 2M_1h_\delta(s_n)$$

The proof for (b) runs in similar way.

Remark 3.5. (a) The growth condition on Q near zero is equivalent to the condition that $\int_0^1 x^\varepsilon\ d\nu(x)=\infty$ for some $\varepsilon\in(0,1)$.

(b) The growth condition on Q near infinity is equivalent to the condition that $\int_1^\infty x^\varepsilon\ d\nu(x)<\infty$ for some $\varepsilon\in(0,1)$.

Proof. First we note that for $0<a<b$ and $\lambda\in(0,1)$.

$$\int_a^b(1-\lambda)x^{\lambda-1}H(x)\ dx=\int_{a<y\leq b}y^\lambda d\nu(y)+a^\lambda H(a)-b^\lambda H(b) \tag{3.7}$$

and

$$\int_a^b\lambda x^{\lambda-1}G(x)\ dx=\int_{a<y\leq b}y^\lambda d\nu(y)+b^\lambda G(b)-a^\lambda G(a) \tag{3.8}$$

Therefore we obtain that for $\lambda\in(0,1)$,

$$\int_0^1 x^\lambda d\nu(x)<\infty \text{ iff } \int_0^1 x^{\lambda-1}Q(x)dx<\infty$$

and

$$\int_1^\infty x^\lambda d\nu(x)<\infty \text{ iff } \int_1^\infty x^{\lambda-1}Q(x)dx<\infty$$

To prove (a), suppose that $\int_0^1 x^\varepsilon d\nu(x)<\infty$ for all $\varepsilon\in(0,1)$.

Then $x^\varepsilon Q(x)\leq\varepsilon\int_0^x y^{\varepsilon-1}Q(y)dy\longrightarrow 0$ as $x\longrightarrow 0$.

This contradicts to the growth condition on Q near zero.

Conversely, if $x^\varepsilon Q(x)\geq 1$ for x sufficiently small, say $x\leq a$, then

$$\begin{aligned}\int_0^1 x^{\varepsilon/2-1}Q(x)dx&\geq\int_0^a x^{-(1+\varepsilon/2)}x^\varepsilon Q(x)dx\\&\geq\int_0^a x^{-(1+\varepsilon/2)}dx=\infty\end{aligned}$$

The proof of (b) is similar. If $\int_{x>1}x^\varepsilon d\nu(x)<\infty$ for some $\varepsilon\in(0,1)$ then

$$x^\varepsilon Q(x)\leq(1-\varepsilon)\int_x^\infty y^{\varepsilon-1}Q(y)dy\longrightarrow 0 \text{ as } x\longrightarrow 0.$$

which gives the growth condition on Q near infinity.

Conversely, if $x^{\varepsilon}Q(x)\leq 1$ for x sufficiently large, then

$$\int_1^{\infty} x^{\varepsilon/2-1}Q(x)\,dx<\infty$$

This completes the proof.

Remark 3.6. If lim sup $G(x)/H(x)<\theta<\infty$ as $t\longrightarrow 0$ and $t\longrightarrow\infty$ then the growth condition on Q near zero and infinity is satisfied, and (3.1) holds for each $\delta>0$. To prove this, we first note that if $0<a<b$ then by (3.7) and (3.8)

$$b^{\lambda}Q(b)-a^{\lambda}Q(a)=\int_a^b x^{\lambda-1}\{\lambda\ G(x)-(1-\lambda)\ H(x)\}\,dx$$

Thus if lim sup $G(x)/H(x)<\theta<\infty$ then $Q(x)$ is strictly decreasing for x sufficiently small and large where $\lambda=(1+\theta)^{-1}$, which gives the growth condition on Q. And if $0<\delta_1<\delta_2$ then

$$(\delta_2/\delta_1)^{-1/\lambda}\leq a_{\delta_2}(t)/a_{\delta_1}(t)<1$$

Hence (3.1) holds for all $\delta>0$.

References

[1] R.M. Blumenthal and R.K. Getoor, Sample functions of stochastic processes with stationary independent increments, *J. Math. Mech.* ***10*** (1961), 493~516.

[2] W. Feller, An extension of the law of the iterated logarithm without varience, *J. Math. Mech.* ***18*** (1968), 343~355.

[3] B.E. Fristedt, The behavior of increasing stable processes for both small and large times, *J. Math. Mech.* ***13*** (1964), 849~856.

[4] B.E. Fristedt, Sample function behavior of increasing processes with stationary independent increments, *Pacific J. Math.* ***21*** (1967), 21~33.

[5] B.E. Fristedt, Sample functions of stochastic processes with stationary independent increments, Advances in probability, vol.3, Marcel Dekker, New York (1974), 241~396.

[6] B.E. Fristedt, and W.E. Pruitt, Lower functions of increasing random walks and subordinators, *Z. Wahrsch. verw. Geb.* ***18*** (1971), 167~182.

[7] N.C. Jain and W.E. Pruitt, Lower tail probability estimates for subordinators and nondecreasing random walks, *Ann. prob.* ***15*** (1987), 75~101.

[8] Y.K. Kim and I.S. Wee, General laws of the iterated logarithm for Levy processes, *J. Korean statist Soc.* ***17*** (1988), 30~45.

[9] Y.K. Kim and I.S. Wee, Upper functions for Levy processes having only negative jumps, Stochastic Analysis and Appl. (1990), to appear.

[10] S. Kochen and C. Stone. A note on the Borel-Cantelli lemma, Illinois *J. Math.* **8** (1964), 248~251.

[11] W.E. Pruitt, General one-sided laws of the iterated logarithm, *Ann. Prob.* **9** (1981), 1~48.

[12] I.S. Wee, Lower functions for processes with stationary independent increments, *Probab. Th. Rel. Fields* **77** (1988), 551~566.