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A CONDITION OF THE WEAK-RESTRICTED LIE ALGEBRA TO BE RESTRICTED

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1. Introduction

The concept of restricted Lie algebra was introduced by Jacobson in 1937. We recall that a restricted Lie algebra over an algebraically closed field F of prime characteristic $p > 0$, is a Lie algebra with a mapping $x \rightarrow x^p$ satisfying certain conditions. Various restricted Lie algebras were found and the classification of the restricted Lie algebras has been elaborated in several ways. In connection with the classification problem of the restricted Lie algebras, the investigation of the tori and the Cartan subalgebras was naturally established. As we know, the tori and Cartan subalgebras are closely related and they play the important role in the classification problem of those. Jacobson and Winter [4,6], more than twenty years ago, made several relations between them. In this paper we will introduce the notion of the weak-restricted Lie algebra L, which is generalized concept of restricted Lie algebra, and Benkart's exponential mapping E^z. And we study some results related to the image of L and their Cartan subalgebras under the mapping E^z. Throughout this paper we assume that L is a finite-dimensional Lie algebra over an algebraically closed field of prime characteristic $p > 0$.

2. Preliminaries

Suppose that L is a restricted Lie algebra $x \in L$ and $M = \overline{<x>}$. Then M is commutative restricted subalgebra of L. An element of M will be called p-polynomial in x. Necessarily there exists unique integer n

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such that $\{x^{p^i} \mid 0 \leq i \leq n\}$ is a basis of M and unique scalars a_i $0 \leq i < n$ with

$$x^{p^n} = \sum_{i=0}^{n-1} a_i x^{p^i}.$$

This equation will be called minimal p-equation for x.

As we know, the semi-simple and nilpotent part of the Jordan-Chevalley decomposition of x are represented by p-polynomials for x.

We now define some Lie algebra that is a generalized concept of restricted Lie algebras.

DEFINITION 2.1. *Suppose H is a Cartan subalgebra of L and $L = H + \sum_\alpha L_\alpha$ is the root space decomposition with respect to H. Then L is called weak-restriced with respect to H if;*

a) H is restricted

b) $(adh)^p x = [h^p, x]$ for $h \in H,\ x \in L$

c) For each $x \in L$, the semisimple component of adx is equal to ads for some semisimple elements $s \in H$

d) For any $x \in L$, if $x = s + n$ is the Jordan-Chevalley decomposition then $s \in L$.

For a semisimple Lie algebra L, we can identify L with $ad\,L \subset Der\,L$ via adjoint mapping, we take a modification of the definition c); for each root α with $x \in L_\alpha$, the semisimple component of x is equal to s for some semisimple elements $s \in H$. Furthermore the decomposition $x = s + n$ is unique. As a corollary of Schue [5] (Proposition 1.6) we have

PROPOSITION 2.1. *If L is restricted and H is a Cartan subalgebra, then L is weak-restricted with respect to H.*

The converse of the proposition is not true. For example, $W(1:2)$ is not restricted but it is weak-restricted with respect to the Cartan subalgebra $H = < xD, x^{p+1}D, \cdots, x^{p(p-1)+1}D >$.

Note 2.1. Let L be a weak-restricred Lie algebra with respect to a Cartan subalgebra H. For every $x \in L$ there exists $k \in N$ such that x^{p^k} is semisimple.

3. Benkart's switching method

Let H be a torus of L and $K = \sum_\alpha L_\alpha$ is the root space decomposition with respect to H. Let z be a nonzero root vector of L_θ, $\theta \neq 0$, relative to H and define the truncated exponential mapping

$$expz = \sum_{j=0}^{p-1} \frac{(adz)^j}{j!}.$$

Let $\operatorname{Ker}\theta$ denote the kernel of θ in H. Since $\operatorname{Ker}\theta$ is a torus, it has a toral basis $\{h_1, \cdots, h_m\}$. This basis may be extended to a toral basis $\{h_1, \cdots, h_m, t\}$ of H ([4] Theorem V.13). Since $t^p = t$, we have $(adt)^p = adt$, impling $\theta(t) \in Z_p$. Thus by adjusting by a scalar in Z_p, if necessary, we may further suppose that $\theta(t) = -1$. Hence $expzH = \operatorname{Ker}\theta \oplus <t+z>$ and

$$(t+z)^{p^r} = t + z + z^p + \cdots + z^{p^r}$$

for each integer $r \geq 0$. We form the abelian restricted subalgebra $S = \overline{expzH} = \operatorname{Ker}\theta \oplus \overline{<t+z>}$ and consider the chain $S \supset S^p \supset S^{p^2} \cdots$.

By [6] Proposition 2.11 $S^{p^{r-1}}$ is a torus for some integer $r \geq 1$ and hence by [6] Proposition 2.5 $S^{p^{r-1}} = S^{p^r} = \cdots$.

Benkart [1] elaborated with some results concerning about the tori. For each non-zero vector $z \in L_\theta$ for $\theta \neq 0$, there is a unique minimal degree p-polynomial $f_z(\lambda)$ such that $f_z(z) \in \operatorname{Ker}\theta$. In fact, if

$$f_z(\lambda) = \lambda^{p^r} + a_{r-1}\lambda^{p^{r-1}} + \cdots + a_k\lambda^{p^k}$$

where $a_k \neq 0$, is a monic p-polynomial of least degree r such that $f_z(z) \in \operatorname{Ker}\theta$, then k is the least integer such that $S^{p^{k-1}} = S^{p^k}$. Furthermore

$$S^{p^k} = <h_1, \cdots, h_m, (t+z)^{p^k}, \cdots, (t+z)^{p^r}>$$

is the maximal torus in $S = \overline{expzH}$. Then S^{p^k} is the maximal torus of S [6]. Winter [6] was the first to develop the technique for producing other tori from a given torus. In the following we adopt his notation $e^z H$ for the maximal torus S^{p^k} in $S = \overline{expzH}$. Benkart defined the following transformation on $\mathrm{Ker}(adz)^{p^k}$, which is called the generalized exponential mapping

$$E^z = 1 + \sum_{j=0}^{p-1} (\Pi_{\nu=1}^{j} (\nu \cdot 1 - \psi)^{-1})(adz)^j$$

where $\psi = (adz)^p + \cdots + (adz)^{p^{k-1}}$. He proved following

THEOREM 3.1. *Let L be a finite-dimensional Lie algebra over an algebraically closed field of prime characteristic and H be a torus of L. If $z \in L_\theta$ is a root vector relative to H, then*

a) there is a unique monic p-polynomial of least degree

$$f_z(\lambda) = \lambda^{p^r} + a_{r-1}\lambda^{p^{r-1}} + \cdots + a_k \lambda^{p^k}$$

where $a_k \neq 0$ such that $f_z(z) \in Ker\theta$.

b) $e^z H = \overline{Ker\theta \oplus < (t+z)^{p^k}, \cdots, (t+z)^{p^r} >}$ is the unique maximal torus of $S = \overline{expzH} = \overline{< t+z >}$ (Here $t^p = t$ and $\theta(t) = -1$)

c) $\dim e^z H = \dim H + r - k$

d) $Z_L(e^z H) = \{E^z y \mid y \in Z_L(H) \cap Ker(adz)^{p^k}\}$

e) $\dim Z_L(e^z H) = \dim(Z_L(H) \cap Ker(adz)^{p^k})$.

4. Main Theorem

In this section, we assume that L is a weak-restricted simple Lie algebra with respect to a two dimensional toral Cartan subalgebra H. Let $\{h, t\}$ be a toral basis of H and $\mathrm{Ker}\,\theta = Fh$ and $L = H + \sum_\alpha L_\alpha$ be the Cartan decomposition with respect to H. Then by Lemma 2.1 $z^{p^l} \in H$ for sufficiently large l and $\theta(z^{p^l}) = 0$ by [3] Lemma 1.4.5. Therefore $z^{p^l} \in \mathrm{Ker}\,\theta$. Now the minimal polynomial of least degree $f_z(\lambda)$, for which $f_z(z) \in \mathrm{Ker}\,\theta$, is λ^{p^l}. Then the unique maximal torus $e^z H$ of $S = \overline{expzH}$ is

$$< \mathrm{Ker}\,\theta, (t+z)^{p^l} > = < h, (t+z)^{p^l} >$$

by Theorem 3.1.b). Now $Z_L(e^z H) \supset <h, t+z>$ and h and $t+z$ are linearly independent. Then Theorem 3.1.e) forces $\dim Z_L(e^z H) = \dim H$. Consequently $Z_L(e^z H) = <h, t+z>$.

LEMMA 4.1. *Let L, H and z be as above, then $Z_L(e^z H)$ is a Cartan subalgebra.*

Proof. Clearly $Z_L(e^z H)$ is nilpotent. For any root vector $x \in L_\alpha$, let $[x,h]$ and $[x,t+z]$ be contained in $Z_L(e^z H)$. Suppose that $\alpha(h) \neq 0$, then $[h,x] = \alpha(h)x \in Z_L(e^z H)$ implies $x \in Z_L(e^z H)$. Suppose that $\alpha(h) = 0$, then $\alpha(t) \neq 0$. Therefore since $[t+z,x] = \alpha(t)x + [z,x] \in Z_L(e^z H)$, we have

$$[(t+z)^{p^l}, (\alpha(t)x + [z,x])] = 0.$$

Then

$$\begin{aligned} 0 &= [(t+z)^{p^l}, (\alpha(t)x + [z,x])] \\ &= [(t + z + z^p + \cdots + z^{p^l}), (\alpha(t)x + [z,x])] \\ &= (\alpha(t)^2 x + \alpha(t)[z^p, x] + \cdots + \alpha(t)[z^{p^l}, x]) \\ &\quad + ([z[z,x]]) \\ &\quad + (2\alpha(t)[z,x] - [z,x] + [z^p[z,x]] + \cdots + [z^{p^l}[z,x]]). \end{aligned}$$

Since the first, second and third parentheses of the last equality are contained in L_α, $L_{\alpha+2\theta}$, and $L_{\alpha+\theta}$ respectively, they are linearly independent. So $[z[z,x]] = 0$ and $[z^{p^i}, x] = 0$, $i = 1, \cdots, l$. Thus we have $x = 0$ since $\alpha(t) \neq 0$. Then we have that $Z_L(e^z H)$ is selfnormalizing.

Our main Theorem is;

THEOREM 4.1. *Let L be a finite-dimensional simpel Lie algebra over an algebraically closed field F of characteristic $p > 0$, having a two dimensional toral Cartan subalgebra H. If L is weak-restricted with respect to H, then L is restricted.*

Proof. Let H, θ and z be as above. Then $(t+z)^{p^l}$ is semi-simple since $e^z H$ is torus, therefore

$$t + z = (t+z)^{p^l} + ((t+z) - (t+z)^{p^l})$$

is the Jordan-Chevalley decomposition by [5] (Theorem 1). So $(t+z)^{p^l} \in L$ since L is weak-restricted and $(t+z)^{p^l} \in Z_L(e^z H)$.

Therefore $Z_L(e^z H) = < h, (t+z)^{p^l} >$ since h and $(t+z)^{p^l}$ are linearly independent. Then $Z_L(e^z H) (= e^z H)$ is a torus and $Z_L(e^z H) = < h, t+z >$ forces $t+z$ is semi-simple and $Z_L(e^z H)$ is restricted. Thus $(t+z)^p = t + z + z^p \in Z_L(e^z H) \subset L$. So z^p is contained in L and consequently L is restricted.

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