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MODIFIED CHAIN CONDITIONS FOR N-MODULES

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1. Introduction

A near-ring N is a system $(N,+,\cdot)$ such that $(N,+)$ is a group, $(N,\cdot)$ is a semigroup, $(a+b)c = ac+bc$ for every a,b,c in N. An N-module M is a system ${}_NM$ such that M is an additive group admitting scalar multiplication by the element of N with the properties : $(a+b)x = ax+bx$, $(ab)x = a(bx)$ for each a, b in N and x in M. An N-group M is a near-ring module ${}_NM$ with the property that $a(x+y) = ax+ay$ for all a in N and x, y in M. Obviously, every distributive near-ring is an N-group.

A nonempty subset A of M is an (resp. normal) N-subgroup of M, if A is an additive (resp. normal) subgroup of M and $NA = \{ra \mid r \in N, a \in A\} \subset A$. A is an N-submodule of M, if $(A,+)$ is normal subgroup of $(M,+)$ and if $r(x+a)-rx$ is in A for every r in N, x in M and a in A. Every N-submodule is an N-subgroup, but not conversely. A nonempty subset B of N is an ideal of N if it is N-submodule of ${}_NN$ and $BN \subset B$. Every N-subgroup of an N-module is also an N-module.

In this paper, we define the concepts of almost descending chain condition for near-ring modules (DCCN) and almost ascending chain condition of near-ring modules (ACCN) which are more generalized concepts of DCCN and ACCN of near-ring modules, and investigate the basic properties of near-ring modules with almost DCCN and those of near-ring modules with almost ACCN.

Similarly we can also define the concepts of almost DCCI, almost ACCI, of near-ring modules, and those of almost DCCL, almost ACCL, almost DCCR and almost ACCR of near-rings. We will see that some of the well known properties of near-ring module with DCCN (ACCN, resp.) also hold for near-ring module with almost DCCN (ACCN, resp.).

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For s-unital near-ring modules, the concept of almost DCCN (ACCN, resp.) will be equivalent to that of DCCN (ACCN, resp.).

C. Faith [3], I. N. Herstein and L. Small [5], B. Johns [4] studied chain conditions on principal annihilator left ideals and on principal left ideals of ring with identity. We will study chain conditions on principal annihilator left ideals and on principal N-subgroups of left s-unital near-rings and investigate relationships between left κ-regularity and DCCN for principal N-subgroups, between right κ-regularity and ACCL for principal annihilator left ideals.

2. Modified Chain Conditions for Near-Ring Modules

First, we will introduce the notions of almost DCCN and almost ACCN for near-ring modules.

Throughout the present paper, N and M will represent a near-ring and near-ring module respectively. For a left ideal I of N and an N-subgroup M' of M,

$$I^{-i}M' := \{x \in M | I^i x \subset M'\},$$

for all positive integer i, we see that

$$M' \subset I^{-1}M' \subset I^{-2}M' \subset \cdots \subset I^{-i}M' \subset \cdots .$$

DEFINITION 2.1. *Let M be an N-module.*

(1) If for each descending chain $M_1 \supset M_2 \supset \cdots$ of N-subgroups of M, there exist positive integers q, m such that $N^q M_m \subset M_i$ for all $i \in Z^+$, then we call that M has almost DCCN.

(2) If for each ascending chain $M_1 \subset M_2 \subset \cdots$ of N-subgroups of M, there exist positive integers, q, m such that $M_i \subset N^{-q}M_m$ for all $i \in Z^+$, then we call that M has almost ACCN.

REMARK 2.2. *Let M be an N-module.*

(1) If M has DCCN, then M has almost DCCN.

(2) If M has ACCN, then M has almost ACCN.

But the converse does not hold as in following examples.

Proof. (1) Suppose that M has DCCN.

Let $M_1 \supset M_2 \supset \cdots$ be a descending chain of N-subgroups of M, since M has DCCN, there exists $p \in Z^+$ such that $M_p \subset M_i$ for all $i \in Z^+$, for given p, $N^p M_p \subset NM_p \subset M_p \subset M_i$, for all $i \in Z^+$. Hence M has almost DCCN.

(2) Suppose that M has ACCN.
Let $M_1 \subset M_2 \subset \cdots$ be an assending chain of N-subgroups of M, since M has ACCN, there exists $m \in Z^+$ such that $M_m = M_{m+1} = \cdots$, that is, $M_i \subset M_m$ for all $i \in Z^+$, for given m, $N^m M_i \subset NM_i \subset NM_m \subset M_m$, that is, $M_i \subset N^{-m} M_m$, for all $i \in Z^+$.
Hence M has almost ACCN.

DEFINITION 2.3. *If ${}_N N$ has almost DCCN (almost ACCN, resp.) then we say that N has almost DCCN (almost ACCN, resp.).*

EXAMPLES 2.4. *(1) If M is any N-module with trivial multiplication (i.e., $NM = 0$), then M satisfies both almost DCCN and almost ACCN.*

(2) If N is any near-ring with trivial multiplication, then N has both almost DCCN and almost ACCN. There are several types of trivial multiplications of near-ring, see. G, Pilz [8], examples 1.4 (b).

(3) Every nilpotent near-ring has both almost DCCN and almost ACCN. For concrete examples, $N = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$ is nilpotent near-ring which has almost DCCN but not DCCN.

Similarly for $N = \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix}$.

(4) $N = \begin{pmatrix} Q & 0 \\ Q & 0 \end{pmatrix}$ is a near-ring with almost ACCN but not ACCN.

(5) $N = \begin{pmatrix} Z & 0 \\ Q & 0 \end{pmatrix}$ is a near-ring with almost ACCN but not ACCN, not DCCN, and not almost DCCN.

(6) The p-adic group Z_{p^∞} with trivial multiplication is a near-ring, we see that Z_{p^∞} has almost DCCN but not has DCCN.

THEOREM 2.5. *The following statements are equivalent.*

(1) M has almost DCCN.

(2) For each descending chain $M_1 \supset M_2 \supset \cdots$ of N-subgroups of M, there exists $p \in Z^+$ such that $N^p M_p \subset M_i$, for all $i \in Z^+$.

(3) Every N-subgroup of M has almost DCCN.

(4) For each non-empty family μ of N-subgroups of M, there exists an element K of μ and a positive integer p such that $N^pK \subset J$, for any J in μ satisfying $J \subset K$.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). These are easily proved. We need to show that (1) $\Longleftrightarrow$ (4). The condition implies almost DCCN.

Suppose M has almost DCCN and μ is a non empty family of N-subgroups of M for which at the condition does not hold. Choose any $K_1 \in \mu$, then there is an $K_2 \in \mu$ such that $K_1 \supset K_2$ but $NK_1 \not\subset K_2$, there is an $K_3 \in \mu$ such that $K_2 \supset K_3$ but $N^2K_2 \not\subset K_3 \cdots$. There is an element K_n in μ such that $K_{n-1} \supset K_n$ but $N^{n-1}K_{n-1} \not\subset K_n$ and so forth. The descending sequence $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ violates the hypothesis that M has almost DCCN.

THEOREM 2.6. *The following statements are equivalent.*

(1) M has almost ACCN.

(2) For each ascending chain $M_1 \subset M_2 \subset \cdots$ of N-subgroups of M, there exists $p \in Z^+$ such that $N^pM_i \subset M_p$, for all $i \in Z^+$.

(3) Every N-subgroup of M has almost ACCN.

(4) For each non-empty family μ of N-subgroups of M, there exists an element K of μ and $p \in Z^+$ such that $N^pJ \subset K$ for any J in μ satisfying $K \subset J$.

Proof. (1) $\Longleftrightarrow$ (4). The condition implies almost ACCN. Suppose that M has almost ACCN and μ is a nonempty family of N-subgroups of M for which the condition does not hold. Pick any $K_1 \in \mu$. Then there is an $K_2 \in \mu$ such that $K_1 \subset K_2$ but $N^2K_2 \not\subset K_1$, there exists an $K_3 \in \mu$ such that $K_2 \subset K_3$ but $N^3K_3 \not\subset K_2 \cdots$. There is an element K_n in μ such that $K_{n-1} \subset K_m$ but $N^nK_n \not\subset K_{n-1}$ and so forth. The ascending sequence $K_1 \subset K_2 \subset \cdots K_n \subset \cdots$ violates the hypothesis that M has almost ACCN.

(1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) is left to the reader.

We will show that these properties are invariant under N-module homomorphisms.

LEMMA 2.7. *Let M_1 and M_2 be two N-modules.*

(1) If M_1 has almost DCCN and $f : M_1 \to M_2$ is epimorphism then M_2 has also almost DCCN.

(2) If M_1 has almost ACCN and $f : M_1 \to M_2$ is epimorphism then M_2 has also almost ACCN.

Proof. (1) Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be a descending chain of N-subgroups of M_2, then

$$f^{-1}(K_1) \supset f^{-1}(K_2) \supset f^{-1}(K_3) \cdots$$

is a descending chain of N-subgroups of M_1. Since M_1 has almost DCCN, there exists a positive integer p such that

$$N^p f^{-1}(K_p) \subset f^{-1}(K_i)$$

for all positive integer i. Since f is epimorphism,

$$N^p K_p = N^p f f^{-1}(K_p) = f[N^p f^{-1}(K_p)] \subset f f^{-1}(K_i) = K_i$$

for all positive integer i.

(2) Let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be an ascending chain of N-subgroups of M_2, then

$$f^{-1}(K_1) \subset f^{-1}(K_2) \subset f^{-1}(K_3) \subset \cdots$$

is an ascending chain of N-subgroups of M_1. Since M_1 has almost ACCN, there exists a positive integer p such that

$$N^p f^{-1}(K_i) \subset f^{-1}(K_p)$$

for all positive integer i. Since f is epimorphism,

$$N^p K_i = N^p f f^{-1}(K_i) = f[N^p f^{-1}(K^i)] \subset f f^{-1}(K_p) = K_p,$$

for all positive integer i.

THEOREM 2.8. *Let N be zero-symmetric and M an N-group and let I be an N-submodule of M. Then*

(1) M has almost DCCN if and only if I and M/I have both almost DCCN.

(2) M has almost ACCN if and only if I and M/I have both almost ACCN.

Proof. The only if parts of (1) and (2) follows from Theorem 2.5, 2.6 and Lemma 2.7. We will show that the if parts of (1) and (2).

(1) Let I and M/I have almost DCCN and let

$$K_1 \supset K_2 \supset K_3 \supset \cdots$$

be a descending chain of N-subgroups of M. Then

$$K_1 \cap I \supset K_2 \cap I \supset K_3 \cap I \supset \cdots$$

is a descending chain of N-subgroups of I, and

$$(K_1 + I)/I \supset (K_2 + I)/I \supset (K_3 + I)/I \supset \cdots$$

is a descending chain of N-subgroups of M/I.
So there is an index $p \in Z^+$ such that

$$N^p[(K_p + I)/I] \subset (K_i + I)/I$$

and

$$N^p(K_p \cap I) \subset K_i \cap I.$$

Take any $j \in Z^+$ and any $a_1, a_2, \cdots a_p \in N$ and any $x \in K_p$. Then

$$a_1 a_2 \cdots a_p(x + I) \subset N^p(K_p + I) \subset K_{p+j} + I.$$

Say $k \in K_{p+j}$, $s \in I$ such that $a_1 a_2 \cdots a_p x = k + s$. But $s \in K_p$ and so $s \in K_p \cap I$. Thus

$$N_s^p \subset N^p(K_p \cap I) \subset K_{p+j} \cap I.$$

Finally,

$$N^p a_1 a_2 \cdots a_p x = N^p(k + s) \subset N_k^p + N_s^p \subset K_{p+j}.$$

So $N^{2p}x \subset K_{p+j}$ that is $N^{2p}K_p \subset K_{p+j}$. Since $j \in Z^+$ was arbitrary $N^{2p}K_p \subset K_i$ for each $i \geq p$. By theorem 2.5, M has almost DCCN.

(2) Let I and M/I have both almost ACCN and let

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

be an ascending chain of N-subgroups of M. Then

$$K_1 \cap I \subset K_2 \cap I \subset K_3 \cap I \subset \cdots$$

is an ascending chain of N-subgroups of I and

$$(K_1 + I)/I \subset (K_2 + I)/I \subset (K_3 + I)/ \subset \cdots$$

is an ascending chain of N-subgroups of M/I. So there is an $p \in Z^+$ such that

$$N^p[(K_i + I)/I] \subset (K_p + I)/I$$

and

$$N^p(K_i \cap I) \subset K_p \cap I.$$

Take any $j \in Z^+$ and any $a_1, \ldots, a_p \in N$ and any $x \in K_{p+j}$. Then

$$a_1 \cdots a_p(x + I) \subset N^p(K_{p+j} + I) \subset K_p + I.$$

Say $k \in K_p$, $s \in N$ such that $a_1 \cdots a_p x = k + s$. But $s \in K_{p+j}$ and so $s \in K_{p+j} \cap I$. Thus

$$N_s^p \subset N^p(K_{p+j} \cap I) \subset K_p \cap I.$$

Finally,

$$N^p a_1 \cdots a_p x \subset N_k^p + N_s^p \subset K_p.$$

So $N^{2p}x \subset K_p$ that is $N^{2p}K_{p+j} \subset K_p$. But $j \in Z^+$ was arbitrary, so $N^{2p}K_i \subset K_p$ for all $i \geq p$. From Theorem 2.6, M has almost ACCN.

THEOREM 2.9. *Let $N = N_0$ and let M be an N-module*

(1) If I is N-submodule of M and M has almost DCCN. Then M/I and I have both almost DCCN.

(2) If I is N-submodule of M and M has almost ACCN. Then M/I and I have both almost ACCN.

Proof. (1) By Lemma 2.7 and Theorem 2.5.

(2) By Lemma 2.7 and Theorem 2.6.

THEOREM 2.10. *(1) A finite product of N-module has almost DCCN if and only if each factor module has almost DCCN.*

(2) A finite product of N-module has almost ACCN if and only if each factor module has almost ACCN.

Proof. By Lemma 2.7.

DEFINITION 2.11. *Let M be an N-module. M is called S-unital if $x \in Nx$ for each $x \in M$. In particular, if N is s-unital then N is called a left S-unital near-ring.*

If N is a near-ring with left identity then N is left S-unital.

REMARK 2.12. *If M is an s-unital module then $M' = NM'$, for each N-subgroup M' of M.*

A non-zero N-module is called irreducible (simple, resp.) if it does not contain any proper N-subgroup (N-submodule, resp.).

THEOREM 2.13. *Let M be an S-unital N-module.*

(1) M has DCCN if and only if M has almost DCCN.

(2) M has ACCN if and only if M has almost ACCN.

Proof. (1). By Remark 2.2. (1).

Conversely, suppose M has almost DCCN. Let $M_1 \supset M_2 \supset \cdots$ be a descending chain of N-subgroups of M. Since M has almost DCCN, there exists a positive integer p such that $N^p M_p \subset M_i$ for all $i \in Z^+$. By Remark 2.12,

$$M_p = NM_p = N^2 M_p = \cdots = N^p M_p \subset M_i$$

for all $i \in Z^+$ that is, there exists a positive integer p such that

$$M_p = M_{p+1} = \cdots .$$

Hence M has DCCN.

(2) By Remark 2.2. (2).

Conversely, suppose M has almost ACCN. Let,

$$M_1 \subset M_2 \subset \cdots$$

be an ascending chain of N-subgroups of M. Since M has almost ACCN, there exists a positive integer p such that $N^p M_i \subset M_p$ for all $i \in Z^+$. By Remark 2.12,

$$M_i = NM_i = N^2 M_i = \cdots = N^p M_i \subset M_p,$$

for all $i \in Z^+$, that is, there is a positive integer p such that

$$M_p = M_{p+1} = \cdots .$$

Hence M has ACCN.

3. Relationship between Chain Conditions and κ-regularity

A near-ring N is called left (right, resp.) κ-regular if for every $a \in N$, there exists $x \in N$ such that $a^n = xa^{n+1}(a^n = a^{n+1}x$, resp.) for some positive integer n.

Hereafter in this section, we will sometimes assume that N is a left S-unital near-ring.

If N is a left S-unital near-ring, for any $a \in N$,

$$Na = \{xa | x \in N\}$$

is called a principal (left) N-subgroup of N. Clearly, for each $a \in N$,

$$Na \supset Na^2 \supset Na^3 \supset \cdots$$

is the descending chain of principal N-subgroups of N.

DEFINITION 3.1. *N safisties the DCC on principal N-subgroups of N if for any $a \in N$, there exists a positive integer n such that $Na^n = Na^{n+1} \cdots$.*

Recall that for N-module M and a subset K of M,

$$L(K) := (0 : K) = \{a \in N | aK = 0\} = ann(K)$$

is (left) annihilator of K in N. Clearly $L(K)$ is a left ideal of N and if $N = N_0$, $L(K)$ is N-subgroup of N.

In particular, for any $a \in N$, $L(a)$ is called a principal annihilator left ideal of N. Obviously, for each $a \in N$,

$$L(a) \subset L(a^2) \subset L(a^3) \subset \cdots$$

is the ascending chain of principal annihilator left ideals of N.

DEFINITION 3.2. *N satisfies the ACC on principal annihilator left ideals of N if for any $a \in N$, there exists a positive integer n such that $L(a^n) = L(a^{n+1}) = \cdots$.*

THEOREM 3.3. *Let N be a near-ring with DCC on principal N-subgroups of N. If there exists an element in N which is not a right zero divisor. Then N is monogenic.*

Proof. Let $a \in N$ such that a is not a zero divisor then a^n is also not a zero divisor for any positive integer n. Indeed, if a^n is a zero divisor then there exists non-zero element x in N such that $xa^n = 0$ that is $xa^{n-1}a = 0$ which implies $xa^{n-1} = 0$, continuing this process $x = 0$ which is contradiction. Since N has DCC on principal N-subgroups of N, there exists a positive integer m, such that

$$Na^m = Na^{m+1} = \cdots .$$

Let $x \in N$, then there is an element y in N such that $xa^m = ya^{m+1}$. Which implies that $(x - ya)a^m = 0$. Since a^m is not a zero divisor, $x = ya$ that is for any $a \in N$,

$$a \in Na \quad \text{that is} \quad Na = N.$$

Therefore N is monogenic.

THEOREM 3.4. *Let N be a left S-unital near-ring. Then N is left κ-regular if and only if N has the DCC on principal N-subgroups of N.*

Proof. Suppose N is left S-unital and left κ-regular. Let $a \in N$. Since N is left S-unital Na is principal N-subgroup generated by a. Let

$$Na \supset Na^2 \supset Na^3 \supset \cdots$$

be a descending chain of the principal N-subgroups of N. Since N is left κ-regular, there exists an element x in N and exists a positive integer n, such that $a^n = xa^{n+1}$. Now,

$$Na^n = Nxa^{n+1} \subset Na^{n+1} = Naa^n \subset Na^n,$$

thus we see that $Na^n = Na^{n+1}$. Similarly we have that $Na^{n+1} = Na^{n+2} = \cdots$. Hence N has the DCC on principal N-subgroups. Conversely, let $a \in N$, suppose $Na^n = Na^{n+1} = \cdots$ for some $n \in Z^+$. Since N is left S-unital $a^n \in Na^n = Na^{n+1}$, then there exists x in N such that $a^n = xa^{n+1}$. Hence N is left κ-regular.

COROLLARY 3.5. *Let N be a near-ring with left indentity. Then, N is left κ-regular if and only if N has the DCC on principal N-subgroups.*

REMARK 3.6. *Let N be a right S-unital near-ring. Then N is right κ-regular if and only if N has the DCC on principal right N-subgroups.*

Proof. Similar method of Theorem 3.4.

THEOREM 3.7. *Let N be any near-ring. If N is right κ-regular then N satisfies the ACC on principal annihilator left ideals.*

Proof. Suppose N is right κ-regular. Let $a \in N$. Since N is right κ-regular, there exists an element x in N and a positive integer n, such that $a^n = a^{n+1}x$. Let

$$L(a) \subset L(a^2) \subset L(a^3) \subset \cdots,$$

be the ascending chain of left annihilators. Then for above positive integer n, we see that

$$L(a^n) = L(a^{n+1}) = L(a^{n+2}) = \cdots.$$

For the first equality, let $t \in L(a^{n+1})$ then $ta^{n+1}x = 0$ implies $ta^{n+1} = 0$. Since $a^n = a^{n+1}x$, $ta^n = 0$. Hence $t \in L(a^n)$. For the second equality, let $s \in L(a^{n+2})$ then $sa^{n+2} = 0$, so

$$sa^{n+2}x = saa^{n+1}x = saa^n = sa^{n+1} = 0.$$

Hence $s \in L(a^{n+1})$. The remainder equalities also hold by mathematical induction.

THEOREM 3.8. *Let N be any near-ring with left identity. If N satisfies the ACC on principal annihilator left ideals and N is left κ-regular then N is right κ-regular.*

Proof. Suppose the conditions are satisfied. Let $a \in N$, and let $L(a^p) = L(a^{p+1}) = \cdots$, for some $p \in Z^+$. Since N is left κ-regular, there is an element x in N and some positive integer m such that $a^m = xa^{m+1}$. Without loss of generality, we may take $m = p =: n$. Thus we have that

$$L(a^n) = L(a^{n+1}) = \cdots ,$$

and $a^n = xa^{n+1}$ for some $n \in Z^+$. Hence

$$(1) \qquad x^k a^n = x^k(xa^{n+1}) = x^{k+1}a^{n+1} \text{ for all } k \in Z^+$$

since $a^{n+1} = aa^n = axa^{n+1}$,

$$(1 - ax)a^{n+1} = 0 \text{ i.e } (1 - ax) \in L(a^{n+1}) = L(a^n).$$

So, $(1 - ax)a^n = 0$ that is $a^n = axa^n$. Again by (1)

$$a^{n+1} = aa^n = aaxa^n = a^2xa^n = a^2x^2a^{n+1}.$$

Then

$$(1 - a^2x^2)a^{n+1} = 0, \quad \text{so}$$
$$(1 - a^2x^2) \in L(a^{n+1}) = L(a^n). \quad \text{Thus we have}$$
$$(1 - a^2x^2)a^n = 0, \quad \text{that is } a^n = a^2x^2a^n.$$

Now, we assume that $a^n = a^ix^ia^n$ for some $i \in Z^+ - \{1,2\}$. Then

$$a^{n+1} = aa^n = aa^ix^ia^n = aa^ix^{i+1}a^{n+1} = a^{i+1}x^{i+1}a^{n+1}$$

by (1). It follows that

$$(1 - a^{i+1}x^{i+1})a^{n+1} = 0.$$

So, $(1 - a^{i+1}x^{i+1}) \in L(a^{n+1}) = L(a^n)$. Thus we see that

$$(1 - a^{i+1}x^{i+1})a^n = 0 \text{ that is } a^n = a^{i+1}x^{i+1}a^n.$$

Hence $a^n = a^{k+1}x^{k+1}a^n$ for all $k \geq 0$. In particular,

$$a^n = a^{n+1}x^{n+1}a^n.$$

If we take that $x^{n+1}a^n = z$ in N, then

$$a^n = a^{n+1}z.$$

Therefore N is right κ-regular.

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