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# POWERS GROUPS AND CROSSED PRODUCT $C^*$-ALGEBRAS

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Let $G$ be a discrete group, $l^2(G)$ the Hilbert space of square summable functions of $G$, and $f_g \in l^2(G)$ the function of $G$ which takes the value one at $g \in G$ and zero elsewhere. Then $\{f_g \mid g \in G\}$ is an orthonormal basis of $l^2(G)$. The left regular representation $U$ of $G$ on $l^2(G)$ is given by $U_g(f_h) = f_{gh}$, $g, h \in G$, and the reduced group $C^*$-algebra of $G$, $C_r^*(G)$, is the $C^*$-subalgebra of $B(l^2(G))$ generated by $\{U_g \mid g \in G\}$. It is well-known that the set of finite linear combinations of $\{U_g \mid g \in G\}$ is a dense $*$-subalgebra of $C_r^*(G)$ and there exists a faithful (normalized) trace $\tau$ on $C_r^*(G)$ characterized by $\tau(U_g) = 0$ if $g \neq e$, and $\tau(U_g) = 1$ if $g = e$. A character $\chi$, a group homomorphism from $G$ to the unit circle, induces a $*$-automorphism $\alpha_\chi$ of $C_r^*(G)$ defined by $\alpha_\chi(U_g) = \chi(g)U_g$. Then we have a crossed product $C^*$-algebra $C_r^*(G)X_{\alpha_\chi}Z$.

DEFINITION [3]. *A group $G$ is a Powers group if the following holds. Given any nonempty finite subset $F \subset G\backslash\{e\}$ and any integer $n \geq 1$, there exist a partition $G = D \cup E$ and elements $g_1, \ldots, g_n \in G$ such that*
*(1) $fD \cap D = \phi$ for any $f \in F$*
*(2) $g_jE \cap g_kE = \phi$ for $j, k \in \{1, \ldots, n\}$ with $j \neq k$.*

Free groups with $n$ generators $F_n$, where $n \geq 2$, are Powers groups and the papers [1,4,5] describe several classes of Powers groups. We list some properties about the Powers group $G$. (For the proof, see [2].)
(a) Any conjugacy class in $G$ other than the identity element is infinite.
(b) $G$ is not amenable.
(c) Any subgroup of $G$ of finite index is a Powers group.

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(d) $C_r^*(G)$ is simple and has a unique trace.

In [6] H.-S.Yin proved many interesting properties about the crossed product $C_r^*$-algebra $C_r^*(G)X_{\alpha_\chi}Z$. One of them is if $\chi(G)$ is an infinite group and $C^*(\mathrm{Ker}\,\chi)$ has a unique trace, then $C_r^*(G)X_{\alpha_\chi}Z$ has a unique trace. And he raised two conjectures, one is if $G$ is a Powers group and $\chi(G)$ is infinite, then $C_r^*(G)X_{\alpha_\chi}Z$ has a unique trace and the other is if $H$ is a normal subgroup of $G$ containing the commutator subgroup of $G$ then $H$ is a Powers group. The first conjecture follows from the second one.

THEOREM 1. *Let $G$ be a Powers group and $H$ a nontrivial normal subgroup of $G$. Then $H$ is a Powers group.*

*Proof.* Let $F$ be a nonempty finite subset of $H\backslash\{e\}$ and an integer $n \geq 1$ be given. Since $G$ is a Powers group, there exist a partition $G = D \cup E$ and elements $g_1, \ldots, g_n \in G$ such that $fD \cap D = \phi$ and $g_jE \cap g_kE = \phi$ for $j \neq k$. Let $D' = D \cap H$ and $E' = E \cap H$, if $n \geq 2$. If $n = 1$, take $D' = \{e\}$ and $E' = H\backslash\{e\}$. We claim that neither $D'$ nor $E'$ is empty. If we assume the contrary, we have either $H \subset D$ or $H \subset E$. Since $H = fH \subset fD \subset E$, for $f$ in $F$, the case $H \subset D$ does not occur. Hence $H \subset E$. Taking $f$ in the set $F$, we have $g_1^{-1}g_2fD \subset g_1^{-1}g_2E \subset D$. Since $g_2^{-1}g_1 \in g_2^{-1}g_1H \subset g_2^{-1}g_1E \subset D$ we then have $g_1^{-1}g_2fg_2^{-1}g_1 \in D$. This is a contradiction since $g_1^{-1}g_2fg_2^{-1}g_1 \in H$. This proves the claim. Now fix $g_1$ and consider the subsets $E, g_1^{-1}g_2E, \ldots, g_1^{-1}g_iE, \ldots, g_1^{-1}g_nE$ which are pairwise disjoint. Then $H = D' \cup E'$ and

(1) $fD' \cap D' \subset fD \cap D = \phi$ for any $f \in F$.

(2) Since $g_1^{-1}g_ifg_i^{-1}g_1E' \subset g_1^{-1}g_ifg_i^{-1}g_1E \subset g_1^{-1}g_iE$ for $i = 2, \ldots, n$, if we set $h_1 = e$ and $h_i = g_1^{-1}g_ifg_i^{-1}g_1$ for $i = 2, \ldots, n$, we have $h_iE' \cap h_jE' = \phi$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$. This completes the proof.

COROLLARY 2. *If $G$ is a Powers group, then $G$ has no nontrivial amenable normal subgroup.*

COROLLARY 3. *If $G$ is a Powers group and $\chi(G)$ is infinite then the crossed product $C^*$-algebra $C_r^*(G)X_{\alpha_\chi}Z$ is simple with a unique trace.*

*Proof.* For the simplicity of $C_r^*(G)X_{\alpha_\chi}Z$, see [3].

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