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# A FUNCTIONAL CENTRAL LIMIT THEOREM FOR ASSOCIATED RANDOM MEASURES

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## 1. Introduction

One of the major problems in the theory of random measures is obtaining an explict expression for Prob $(X(B) \leq x)$, for large regions $B \subset R$, where $X(B)$ represents the mass of the set $B$ for the random measure $X$. Many recent papers have been concerned with various limit theorems for stationary associated random variables (see for example Newman [9], Newman and Wright [10, 11], Burton and Waymire [4], and Burton and Kim [3] etc.)

Moreover Cox and Grimmett (1984) have proved a central limit theorem for associated random variables (which need not to be stationary) subject to certain conditions on the moments and covariance. Birkel (1988) has also obtained an invariance principle for nonstationary associated processes by weakening the assumption of strict stationarity and replacing it by certain conditions on the moments.

In this paper we investigate a central limit theorem, a functional central limit theorem and a scaling limit in the case of nonstationary random measure satisfying a condition of positive dependence called association. A finite collection $\{X_1, \dots, X_n\}$ of random variables is associated if for any two coordinatewise nondecreasing functions $f_1$, $f_2$ on $R^n$ such that $\hat{f}_i = f_i(X_1, \dots, X_n)$ has finite variance for $i = 1, 2$, there holds $\mathrm{Cov}(\hat{f}_1, \hat{f}_2) \geq 0$. An infinite collection is associated if every finite subcollection is associated (Esary, Proschan and Walkup, 1967).

Burton and Waymire (1985) extend this notion to the random measure.

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A random measure $X$ is associated if and only if the family of random variables $\mathcal{F} = \{X(B) : B \text{ a Borel set}\}$ is associated.

A precise statement of central limit theorem (Cox, Grimmett, 1984) and invariance principle (Birkel, 1988) for a sequence of nonstationary associated random variables is given in Section 2. In the nonstationary case a central limit theorem, a classical scaling limit, and a functional central limit theorem for associated random measures is proven in Section 3 and some relationships among them are also studied.

## 2. Preliminaries

Throughout the paper let $\{X_j : j \in N\}$ be a sequence of random variables on some probability space $(\Omega, \mathcal{F}, P)$ with $EX_j = 0$, $EX_j^2 < \infty$. Let

$$S_n = \sum_{j=1}^{n} X_j, \quad \sigma_n^2 = ES_n^2, \quad \text{for} \quad n \in N.$$

$\{X_j : j \in N\}$ is said to satisfy the central limit theorem if $\sigma_n^{-1} S_n \underset{n}{\longrightarrow} N(0,1)$.

Define random elements in $D[0,1]$ endowed with the Skorokhod topology by

$$W_n(t) = \sigma_n^{-1} S_{[nt]}, \quad t \geq 0$$

where $S_0 = 0$, $D[0,1]$ is the set of all functions on $[0,1]$ which have left hand limits and are continuous from the right, $\{X_j : j \in N\}$ fulfills the invariance principle if $W_n$ converges weakly to standard Brownian motion.

THEOREM 2.1.(NEWMAN, WRIGHT). *Let $\{X_j : j \in N\}$ be a strictly stationary sequence of associated random variables with $EX_j = 0$, $EX_j^2 < \infty$. Assume*

$$0 < \sigma^2 = Cov(X_1, X_1) + 2 \sum_{j=2}^{\infty} Cov(X_1, X_j) < \infty. \tag{2.1}$$

*Then $\{X_j : j \in N\}$ fulfills the invariance principle.*

Cox and Grimmett (1984) weakened the assumption of strict stationarity and replaced it by certain conditions on the moments of the random variables. Using the coefficient

$$u(n) = \sup_{k \in N} \sum_{j:|j-k| \geq n} \mathrm{Cov}(X_j, X_k), \; j \in N, \; n \in NU\{0\}. \tag{2.2}$$

They obtained the following central limit theorem :

THEOREM 2.2.(COX, GRIMMETT). *Let $\{X_j : j \in N\}$ be a sequence of associated random variables with $EX_j = 0, EX_j^2 < \infty$. Assume*

$$u(n) \underset{n}{\longrightarrow} 0, \quad u(0) < \infty, \tag{2.3}$$

$$\inf_{j \in N} Var(X_j) > 0, \tag{2.4}$$

$$\sup_{j \in N} E|X|^3 < \infty. \tag{2.5}$$

*Then $\{X_j : j \in N\}$ satisfies the central limit theorem, that is, $\sigma_n^{-1} S_n$ is asymptotically normal distributed.*

Note that for a wide sense stationary sequence of associated random variables condition (2.1) implies

$$u(0) = \sigma^2, \; u(n) = 2 \sum_{j=2}^{\infty} \mathrm{Cov}(X_1, X_j), \; n \in N,$$

and hence (2.3) and (2.4) are automatically satisfied. Therefore in the stationary case Theorem 2.2 is the implicit central limit theorem of Theorem 2.1 except the superfluous third moment condition (2.5).

Birkel (1988) has extended Theorems 2.1 and 2.2 to an invariance principle for nonstationary associated processes by assuming a condition on covariances of the process, namely $\sigma_n^{-2} \mathrm{Cov}(S_{nk}, S_{nl}) \underset{n}{\longrightarrow} \min\{k, l\}$ for $k, l \in N$. This condition is a weak form of stationarity and necessary for the invariance principle. Combining Theorem 1, Remark 2 and Lemma 1 in Birkel (1988) we obtain the following theorems :

THEOREM 2.3.(BIRKEL, 1988). *Let $\{X_j : j \in N\}$ be a sequence of associated random variables with $EX_j = 0$, $EX_j^2 < \infty$. Assume*

$$\sigma_n^{-2}\sigma_{[nt]}^2 \underset{n}{\longrightarrow} t \quad \text{for} \quad t > 0, \tag{2.6}$$

$$\{\sigma_n^{-2}(S_{n+m} - S_m)^2 : m \in N \cup \{0\}, n \in N\} \tag{2.7}$$
*is uniformly integrable*

*Then $\{X_j : j \in N\}$ fulfills the invariance principle.*

THEOREM 2.4.(BIRKEL, 1988). *Let $\{X_j : j \in N\}$ be a sequence of associated random variables with $EX_j = 0$, $EX_j^2 < \infty$. Then the following assertions are equivalent:*
*(i) The condition (2.6) is fulfilled*
*and $\{X_j : j \in N\}$ satisfies the central limit theorem,*
*(ii) $\{X_j : j \in N\}$ fulfills the invariance principle.*

Theorem 2.4 shows that the central limit theorem is an important tool in establishing the invariance principle for associated processes.

We conclude this section with introducing some results for associated random measures in Burton and Waymire (1985).

DEFINITION 2.5. *Let $X$ be a random measure. We say that $X$ satisfies a classical scaling limit if for all disjoint rectangles $A_1, \ldots, A_n$.*

$$\left(\frac{X(KA_1) - E[X(KA_1)]}{\sqrt{K}}, \cdots, \frac{X(KA_n) - E[X(KA_n)]}{\sqrt{K}}\right) \tag{2.8}$$

*converges in distribution ($K \to \infty$) to a multivariate normal with mean vector 0 and diagonal covariance matrix whose diagonal terms are $\sigma^2|A_1|$, $\cdots$, $\sigma^2|A_n|$ (where $|A_i|$ is the Lebesgue measure of $A_i$ for some positive parameter $\sigma^2$).*

THEOREM 2.6.(BURTON, WAYMIRE, 1985). *Suppose that $X$ is a stationary associated point random measure such that $EX^2(B) < \infty$ for* *bounded Borel set B and*

$$(2.9)\quad 0 < \sigma^2 = Cov(X(I), X(I)) + 2\sum_{j=2}^{\infty} Cov(X(I), X(I+j-1) < \infty$$

*where I is the unit interval. Then X satisfies a classical scaling limit with parameter* $\sigma^2$.

REMARK. *Burton and Waymire pointed out without proof that the result of Theorem 2.6 can be actually strengthened to get a functional scaling limit (in the sense of a functional central limit) by application of results in Newman and Wright (1981, 1982) (see Theorm 2.1).*

Let $X_K(t) = X_K((0,t))$ be defined by

$$(2.10)\qquad X_K(t) = (K)^{-1/2}[X(0,Kt] - EX(0,Kt]]$$

for $t > 0$.

Then $X_K$ converges in the Skorokhod topology on the appropriate function space to the Brownian motion as $K \to \infty$ under the conditions of Theorem 2.6.

## 3. Results

Cox and Grimmett (1984) have proved a central limit theorem for nonstationary associated random variables as in Theorem 2.2.

We extend this to random measures as follows:

As we have already mentioned in introduction random measure $X$ is associated if and only if the family of random variables $\mathcal{F} = \{X(B) : B$ a Borel set$\}$ is associated.

To obtain the following central limit theorem for associated random measure, we use the coefficient

(3.1)

$$r(n) = \sup_{k\in N} \sum_{j:|j-k|\geq n} \mathrm{Cov}(X(I+j-1), X(I+k-1)),\ j \in N,\ n \in N\cup\{0\},$$

where $I$ is the unit interval.

THEOREM 3.1. *Let $X$ be an associated random measure with* $\operatorname{Var} X(I+j-1) < \infty$, $j \in N$, *and define* $X_K(t)$ *as in (2.10).*
*Assume that*

$$r(n) \underset{n}{\longrightarrow} 0, \quad r(0) < \infty, \tag{3.2}$$

$$\inf_{j \in N} \operatorname{Var} X(I+j-1) > 0, \tag{3.3}$$

$$\sup_{j \in N} E|X(I+j-1)|^3 < \infty, \tag{3.4}$$

*where $I$ is the unit interval. Then $X$ satisfies a central limit theorem.*

*Proof.* Let $X$ denote a random interval function subject to the conditions of the theorem. Consider the distribution of

$$X_K(t) = (X[0, Kt) - EX[0, Kt))/\sqrt{K} \tag{3.5}$$

as $K \to \infty$. Let $Y_j = X(I+j-1) - EX(I+j-1)$, $j \in N$. Then $\{Y_j : j \in N\}$ is a family of associated random variables with $EY_j = 0$, $EY_j^2 < \infty$ and obviously satisfies (2.3), (2.4) and (2.5) in Theorem 2.2. Thus $\{Y_j : j \in N\}$ satisfies a central limit theorem by Theorem 2.2. Also let $I_k^o = [0, Kt) - [0, [Kt])$ where $[Kt]$ denotes the greatest integer in $Kt$. Then

$$\begin{aligned} X_K(t) &= X([0, Kt)) - EX([0, Kt))/\sqrt{K} \\ &= \sum_{j=1}^{[Kt]} Y_j/\sqrt{K} + [X(I_K^o) - EX(I_K^o)]/\sqrt{K}. \end{aligned} \tag{3.6}$$

Since $\operatorname{Var} X(I_K^o) = O(1)$ as $K \to \infty$, it follows from Chebyshev's inequality that the second term in (3.6) converges in probability to zero as $K \to \infty$. Moreover $[K] \sim K$ as $K \to \infty$ the result follows from Theorem 2.2.

COROLLARY 3.2. *Let $X$ be an associated random measure with* $\operatorname{Var} X(I+j-1) < \infty$, $j \in N$. *If $X$ satisfies the conditions (3.2), (3.3), and (3.4) as in Theorem 3.1. Then $X_K(A)$ converges in law to* $N(0, \sigma^2|A|)$

*distribution for any bounded Borel set $A$ where $|A|$ is a Lebesgue measure of $A$, and*

$$X_K(A) = [X(KA) - EX(KA)]/\sqrt{K}. \tag{3.7}$$

Note that for a wide sense stationary associated random measure condition (2.9) implies

$$r(0) = \sigma^2, r(n) = 2\sum_{j=2}^{\infty} \mathrm{Cov}(X(I),\ X(I+j-1)),\ n \in N, \tag{3.8}$$

and hence (3.2) and (3.3) are automatically satisfied.

Therefore in the stationary case Theorem 3.1 is the implict central limit theorem of Theorem 2.6.

THEOREM 3.3. *Suppose that $X$ is an associated point random measure and satisfies the conditions as in Theorem 3.1. Then $X$ satisfies a classical scaling limit.*

*Proof.* Let $I = (0, 1]$. By corollary 3.2 $X_K(I)$ converges in law to $N(0, \sigma^2)$ distribution as $K \to \infty$. For arbitrary disjoint unit intervals $I_1, \ldots, I_m$ the same considerations may be applied to the random vector $(X_K(I_1), \ldots, X_K(I_m))$ and hence the result follows.

In 1988 Birkel has proved the invariance principle for nonstationary associated random variables (see Theorems 2.3 and 2.4), we extend this to random measure.

Let $S_n = \sum_{j=1}^{n}[X(I+j-1) - EX(I+j-1)]$, and $\sigma_n^2 = ES_n^2$, where $I$ is the unit interval.

THEOREM 3.4. *Let $X$ be an associated random measure with $\mathrm{Var}\, X(I+j-1) < \infty$, $j \in N$, where $I$ is the unit interval.*

*Assume that*

$$\sigma_n^{-2}\sigma_{[nt]}^2 \underset{n}{\longrightarrow} t \quad \text{for} \quad t > 0, \tag{3.9}$$

$$(3.10)\qquad \{\sigma_n^{-2}(S_{n+m}-S_m)^2 : m \in N \cup \{0\},\, n \in N\}$$
*is uniformly integrable.*

*Then $X$ fulfills the functional central limit theorem.*

*Proof.* We consider the distribution of

$$(3.11)\qquad \begin{aligned} X_K(t) &= \{X(0,Kt] - EX(0,Kt]\}/\sqrt{K} \\ &= \sum_{j=1}^{[Kt]}(X(I+j-1) - EX(I+j-1)/\sqrt{K} \\ &\quad + [X([Kt],Kt] - EX([Kt],Kt]]/\sqrt{K} \\ &= \sum_{j=1}^{[Kt]} Y_j/\sqrt{K} + [X([Kt],Kt] - EX([Kt],Kt]]/\sqrt{K} \end{aligned}$$

where $Y_j = X(I+j-1) - EX(I+j-1)$.

Note that $\{Y_j : j \in N\}$ is a sequence of associated random variables with $EY_j = 0$ and $EY_j^2 < \infty$.

Since there exists $n = n(K)$ such that $n \to \infty$ as $K \to \infty$, $\{Y_j : j \in N\}$ satisfies conditions (2.6) and (2.7) from (3.9) and (3.10) and hence $\{Y_j : j \in N\}$ fulfills the functional central limit theorem.

On the other hand the second term on the right hand side of (3.11) $(X([Kt],Kt] - EX[Kt],Kt])/\sqrt{K}$ converges in probability to zero as $K \to \infty$ and hence $X$ also fulfills the functional central limit theorem.

THEOREM 3.5. *Let $X$ be an associated random measure with $\mathrm{Var}\, X(I+j-1) < \infty$, and define $X_K(t)$ as in (2.10). Assume that*

$$(3.12)\qquad r(n) \underset{n}{\longrightarrow} 0, \quad r(0) < \infty,$$

$$(3.13)\qquad \inf_{j \in N} \mathrm{Var} X(I+j-1) > 0,$$

$$(3.14)\qquad \sup_{j \in N} E|X(I+j-1)|^3 < \infty,$$

*and*

(3.15)
$$\sigma_n^{-2}\sigma_{[nt]}^2 \underset{n}{\longrightarrow} t \quad \textit{for} \quad t > 0,$$

*Then $X$ fulfills a functional central limit theorem.*

*Proof.* Let $Y_j = X(I+j-1) - EX(I+j-1),\ j \in N$
Then

(3.16)
$$\begin{aligned} X_K(t) &= \{X((0,Kt]) - EX((0,Kt])\}/\sqrt{K} \\ &= \sum_{j=1}^{[Kt]} \{X(I+j-1) - EX(I+j-1)\}/\sqrt{K} \\ &\qquad + \{X(I_K^o) - EX(I_K^o)\}/\sqrt{K} \\ &= \sum_{j=1}^{[Kt]} Y_j/\sqrt{K} + \{X(I_K^o) - EX(I_K^o)\}/\sqrt{K} \end{aligned}$$

where $I_k^o = ((0,Kt]\,(0,[Kt]])$.

Since the second term in (3.16) converges in probability to zero as $K \to \infty$ by Chebyshev's inequality $\{Y_j : j \in N\}$ fulfills a central limit theorem from (3.12), (3.13) and (3.14) and hence by Theorem 2.4 $\{Y_j : j \in N\}$ fulfills a functional central limit theorem because $\{Y_j : j \in N\}$ satisfies condition (2.6) from (3.15).

Thus $X_K$ also fulfills a functional central limit theorem since $\{Y_j : j \in N\}$ fulfills a functional central limit theorem and the second term in (3.16) converges in probability to zero as $K \to \infty$.

The following theorem is an improvement of Theorem 2.4 to associated random measures.

THEOREM 3.6. *Let $X$ be an associated random measure with* Var $X(I+j-1) < \infty$, $j \in N$. *The following assertions are equivalent:*

(i) *The condition (3.15) is fulfilled and $X$ satisfies the central limit theorem.*

(ii) *$X$ fulfills the functional central limit theorem.*

*Proof.* Define $X_K(t)$ as in (3.16) of Theorem 3.5 and note that

$$\{X(I_K^o) - EX(I_K^o)\}/\sqrt{K} \xrightarrow{P} 0 \tag{3.17}$$

as $K \to \infty$, where $\xrightarrow{P}$ represents convergence in probability.
(i) $\Longrightarrow$ (ii) :
Since $X$ satisfies a central limit theorem by (3.17) $\{Y_j : j \in N\}$ also satisfies a central limit theorem.

Furthermore condition (3.15) suggests that $\{Y_j : j \in N\}$ fulfills condition (2.6) and hence it fulfills the functional central limit theorem by Theorem 2.4.

Thus by the fact that $\{Y_j : j \in N\}$ fulfills the functional central limit theorem, (3.16) and (3.17) $X$ fulfills a functional central limit theorem.
(ii) $\Longrightarrow$ (i) :
By assumption and (3.17) $\{Y_j : j \in N\}$ fulfills the functional central limit theorem and hence by Theorem 2.4 it also fulfills a condition (2.6) and satisfies a central limit theorem. Thus by the fact that $\{X_j : j \in N\}$ satisfies a central limit theorem and (3.17) $X$ also satisfeis a central limit theorem and fulfills condition (3.15).

Theorem 3.6 shows that the central limit theorem is an important tool in establishing the functional central limit theorem for associated random measures. Hence it is disirable to look for conditions which imply the central limit theorem.

COROLLARY 3.7. *Let $X$ be an associated random measure with $\mathrm{Var}X(I + j - 1) < \infty$. If $X_K$ fulfills the functional central limit theorem then for any bounded Borel set $A$, $X_K(A)$ converges in law to a $N(0, \sigma^2|A|)$ distribution, where $|A|$ is a Lebesgue measure of $A$ and $\sigma^2 = r(0)$ in (3.8).*

*Proof.* By assumption $X$ fulfills the functional central limit theorem and hence $X$ satisfies a central limit theorem. This completes the proof.

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