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LINEAR DISTANCE FUNCTIONS

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1. Introduction

As is well known a metric d on a set M generates a topology as follows: To each $\varepsilon \in \mathbf{R}^+$ and $x \in M$ the (closed) ε-ball $K_\varepsilon(x)$ is defined as $K_\varepsilon(x) := \{z | z \in M,\ d(x,z) \leq \varepsilon\}$ and a subset T of M is called d-open, if to each $x \in T$ there exists $\varepsilon \in \mathbf{R}^+$ with $K_\varepsilon(x) \subseteq T$. It is almost trivial to prove that the set of d-open sets is a topology. Hereby we see that the conditions of a metric are never used, but only the linear property of the range $\mathbf{R}^+ \cup \{0\}$ is essential. In this aspect we consider a function $d : M \times M \to W$ (namely a distance function) where W is a partial ordered set relative to the order $\leq$ with the property that for every ε, δ of W there is one σ in W such that $\sigma \leq \varepsilon,\ \delta$. In this paper we consider a special case where W is a chain with the smallest element. The function $d : M \times M \to W$ should not possese any other properties except that $K_\varepsilon(x)$ is a neighborhood of x itself. The main result of this paper is the necessary and sufficient conditions for the generation of a topological space by this function. In the last paragraph the range W is replaced by $\mathbf{R}^+ \cup \{0\}$ and the equivalence of both functions is clearly represented.

2. Preliminaries

Throughout this paper we denote W as a partial ordered set relative to an order $\leq$ having the following conditions.

(1) W has the smallest element 0.

(2) For every $\varepsilon, \delta \in W \backslash \{0\}$ there is $\sigma \in W \backslash \{0\}$ such that $\sigma \leq \varepsilon,\ \delta$.

DEFINITION 2.1. *(a) Let M be a set. A function $d : M \times M \to W$ is called a distance function.*

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(b) For every $x \in M$ and $\varepsilon \in W\backslash\{0\}$ the set $K^d_\varepsilon(x) := \{z|z \in M, d(x,z) \leq \varepsilon\}$ is called ε-ball of x relative to d.

(c) A subset T of M is called d-open, if to each $x \in T$ there exists $\varepsilon \in W\backslash\{0\}$ such that $K_\varepsilon(x) \subseteq T$.

THEOREM AND DEFINITION 2.2. *(a) Let M be a set and $d : M \times M \to W$ a distance function. Then the set $\mathcal{T}_d := \{T|T \subseteq M,\ T\ d-\text{open}\}$ is a topology on M and is said to be generated by d.*

(b) A topological space $(M, \mathcal{T})$ is called generated by a distance function d, if there is a distance function $d : M \times M \to W$ such that $\mathcal{T} = \mathcal{T}_d$.

DEFINITION 2.3. *Let M be a set. Two distance functions $d, d' : M \times M \to W$ are called equivalent, if $\mathcal{T}_d = \mathcal{T}_{d'}$.*

DEFINITION 2.4. *Let M be a set and $d : M \times M \to W$ a distance function. If for all $x \in M$ and $\varepsilon \in W\backslash\{0\} K_\varepsilon(x)$ is a neighborhood of x, we say d is topological.*

General Note: For a set $\mathcal{M}$ of sets and an arbitrary x we set $\mathcal{M}(x) := \{X|X \in \mathcal{M}, x \in X\}$.

We introduce in the following a very useful and relatively simple lemma by which most theorems are proved throughout the paper.

THEOREM 2.5. *(Criterion for a topological space generated by a topological distance function): Let $(M, \mathcal{T})$ be a topological space, $(W, \leq)$ a partial ordered set and $d : M \times M \to W$ a topological distance function. Then $\mathcal{T} = \mathcal{T}_d$ if and only if for all $x \in M$ the following holds.*

(1) To each $\varepsilon \in W\backslash\{0\}$ there exists $U \in \mathcal{T}(x)$ with $U \subseteq K_\varepsilon(x)$.

(2) To each $U \in \mathcal{T}(x)$ there exists $\varepsilon \in W\backslash\{0\}$ with $K_\varepsilon(x) \subseteq U$.

Proof. " $\longrightarrow$ ": This is trivial by the definition of topological distance function.

" $\longleftarrow$ ": Suppose the assumptions (1) and (2) are satisfied. We show first $\mathcal{T} = \mathcal{T}_d$. " $\subseteq$ ": Let $T \in \mathcal{T}$ and $x \in T$. By (2) there exists one $\varepsilon \in W\backslash\{0\}$ such that $K_\varepsilon(x) \subseteq T$. " $\supseteq$ ": Let $T \in \mathcal{T}_d$ and $x \in T$. By definition of $\mathcal{T}_d$ there exists one $\varepsilon \in W\backslash\{0\}$ with $K_\varepsilon(x) \subseteq T$. By (1) there exists one $U \in \mathcal{T}(x)$ with $U \subseteq K_\varepsilon(x)$, hence $U \subseteq T$. With (1) it follows that d is topological.

3. Linear Distance Functions

DEFINITION 3.1. *Let M be a set and $d : M \times M \to W$ a distance function. If d is topological and W is a chain with the smallest element 0, then we say that d is linear.*

THEOREM 3.2. *Let $(M, \mathcal{T})$ be a topological space generated by a linear distance function d. Then every point of M has a chain as a neighborhood basis.*

Proof. Since d is topological, for every $x \in M$ the set $\{K_\varepsilon(x) | \varepsilon \in W \setminus \{0\}\}$ is obviously a chain relative to $\subseteq$ and neighborhood basis of x.

This is of course a necessary condition of a topological space generated by a linear distance function. But, as we shall see in 3.11, this is not sufficient for the generation of topological spaces by a linear distance function. Accordingly, we require a new concept such that the sufficient condition of those spaces are established.

DEFINITION 3.3. *Let $(M, \mathcal{T})$ be a topological space. (a) Let $x, y \in M$. y is said to be seperated from x, if there exists $V \in \mathcal{T}(x)$ such that $y \notin V$.*

(b) A point $x \in M$ is said to be approximate, if the set $\mathcal{T}(x)$ does not have the smallest element.

(c) A function $f : M \to M$ is called an x-isolation, if the following conditions are satisfied.

(1) Every point seperated from x is associated to a point seperated from $f(x)$.

(2) f is continuous at x.

THEOREM 3.4. *Let $(M, \mathcal{T})$ be a topological space and $d : M \times M \to W$ a linear distance function such that $\mathcal{T} = \mathcal{T}_d$. Then for every two approximate points $x, y \in M$ there exists an x-isolation function $f : M \to M$ with $f(x) = y$.*

Proof. Let x, y be two approximate points. Define a function $f : M \to M$ as follows.

(1) If $z \in M$ is not seperated from x, we put $f(z) = y$.

(2) If $z \in M$ is seperated from x, then we define $f(z)$ successively as follows.

$$\begin{aligned}
&\text{Choose } U \in \mathcal{T}(x) \text{ such that } z \notin U;\\
&\text{Choose } \varepsilon \in W\backslash\{0\} \text{ with } K_\varepsilon(x) \subseteq U;\\
&\text{Choose } V \in \mathcal{T}(y) \text{ with } V \subseteq K_\varepsilon(y);\\
&\text{Choose } V' \in \mathcal{T}(y) \text{ with } V' \subset V;\\
&\text{Choose } f(z) \in V\backslash V'.
\end{aligned}$$

By construction of f $f(x) = y$, for every $z \in M$ which is seperated from x, $f(z)$ is seperated from y. We now have to prove that f is continuous at x. Let $V \in \mathcal{T}(y)$. There exists one $\delta \in W\backslash\{0\}$ with $K_\delta(y) \subseteq V$ and $U \in \mathcal{T}(x)$ with $U \subseteq K_\delta(x)$. We show $f(U) \subseteq V$. Let $z \in U$. If z is not seperated from x, then $f(z) = y \in V$.

Hence let z be seperated from x. Then by construction of f there exists for z one $\varepsilon \in W\backslash\{0\}$, $U' \in \mathcal{T}(x)$ with $z \notin U'$, $K_\varepsilon(x) \subseteq U'$ such that $f(z) \in K_\varepsilon(y)$. We show here $\varepsilon \leq \delta$. Suppose $\delta < \varepsilon$. Then $z \in U \subseteq K_\delta(x) \subseteq K_\varepsilon(x) \subseteq U'$, i.e., $z \in U'$. It is contrary to $z \notin U'$. Therefore $\varepsilon \leq \delta$ and consequently $f(z) \in K_\varepsilon(y) \subseteq K_\delta(y) \subseteq V$.

PROPOSITION 3.5. *Let $(M, \mathcal{T})$ be a topological space. Let $\mathcal{S}$ be a chain as a neighborhood basis of a point x of M. Then $B := \{\dot{S} | S \in \mathcal{S}\}$, where $\dot{T}$ is interior set of T, is an open neighborhood basis of x and also a chain.*

The following Lemma and Definition 3.6 is a useful tool for the proof of the main theorem.

LEMMA AND DEFINITION 3.6. *Let $\mathcal{M}$ be a chain of sets. Let $\overline{\mathcal{M}} := \{\cup\mathcal{S} | \mathcal{S} \subseteq \mathcal{M}\}$. Then $\overline{\mathcal{M}}$ is a chain with $\phi \in \overline{\mathcal{M}}$, $\mathcal{M} \subseteq \overline{\mathcal{M}}$.*

In a topological space $(M, \mathcal{T})$, and $x \in M$ we obtain the following:

(a) If $\mathcal{M}$ is a subchain of $\mathcal{T}$, then $\overline{\mathcal{M}}$ is also a subchain of $\mathcal{T}$.

(b) If $\mathcal{M}$ is a subchain on $\mathcal{T}(x)$, then $\overline{\mathcal{M}}\backslash\{\phi\}$ is a subchain of $\mathcal{T}(x)$.

THEOREM 3.7. *Let $(M, \mathcal{T})$ be a topological space with the following properties.*

(1) Every $x \in M$ has a chain as a neighborhood basis of x.

(2) For every two approximates $x, y \in M$ there exists an x-isolation function $f : M \to M$ with $f(x) = y$.

Then $(M, \mathcal{T})$ will be generated by a linear distance function.

Proof. We consider the following two cases.

Case 1. An approximate point in M dose not exist. We define a linear function $d : M \times M \to \{1,0\}$, $(x,y) \mapsto \begin{cases} 0, & \text{if } y \in \cap\mathcal{T}(x) \\ 1, & \text{otherwise.} \end{cases}$

It is quite simple to see that $\mathcal{T} = \mathcal{T}_d$ and d is just a quasimetric which is trivially linear. (See [2]).

Case 2. There exists an approximate point $e \in M$. In the following we let e be chosen fixed. By the property (1) and 3.2 every $x \in M$ has an open neighborhood basis which is a chain. Hence for every x we choose such a neighborhood basis B_x. By 3.6 (b) $\overline{B}_x \backslash \{\phi\} \subseteq \mathcal{T}(x)$ and is a subchain of $\mathcal{T}(x)$. For every approximate $x \in M$ we choose a x-isolation function f_x with $f_x(x) = e$ and define $\varphi_x : \mathcal{T}(x) \cup \{\phi\} \to \overline{B}_e$, $U \mapsto \cup\{V | V \in B_e,\ f_x(U) \nsubseteq V\}$.

For every non-approximate $x \in M$ we let $\varphi_x : \mathcal{T}(x) \cup \{\phi\} \to \overline{B}_e$, $U \mapsto \begin{cases} \phi, & \text{if } U = \phi \\ \cup B_e, & \text{otherwise} \end{cases}$. Then for all $x \in M$ the following holds:

(i) φ_x reserves the inclusion and $\phi\varphi_x = \phi$.

(ii) For all $U \in \mathcal{T}(x)$ $U\varphi_x \neq \phi$, hence $U\varphi_x \in \overline{B}_e \backslash \{\phi\} \subseteq \mathcal{T}(e)$.

For non-approximate $x \in M$ the assertions (i), (ii) are trivial.

Now let $x \in M$ be approximate. For (i) let $U, U' \in \mathcal{T}(x) \cup \{\phi\}$ with $U' \subseteq U$. Then $f_x(U') \subseteq f_x(U)$, hence $\{V | V \in B_e,\ f_x(U') \nsubseteq V\} \subseteq \{V | V \in B_e, f_x(U) \nsubseteq V\}$. With this $U'\varphi_x \subseteq U\varphi_x$. For (ii) let $U \in \mathcal{T}(x)$. Since x is approximate, there exists one $U' \in \mathcal{T}(x)$ with $U' \subset U$. Hence there exists $z \in U \backslash U'$ whence z is seperated from x and also $f_x(z)$ seperated from $f_x(x) = e$. In other words, there exists $V \in B_e$ with $f_x(z) \notin V$. Hence $f_x(U) \nsubseteq V$. With this $V \subseteq U\varphi_x$, i.e. $U\varphi_x \neq \phi$.

Now let for all $x, y \in M$ $K(x,y) := \cup\{V | V \in B_x, y \notin V\} \in \overline{B}_x$. Then the following holds for all $x, y \in M$:

(iii) $K(x,y) = \phi$ if and only if y is not seperated from x.

(iv) $y \notin K(x,y)$.

(v) If $y \in U$, $U \subseteq B_x$ then $K(x,y) \subseteq U$.

Since (iii) and (iv) are obvious, it remains to prove only (v).

Let $x, y \in M$ and $U \subseteq B_x$ with $y \in U$. Assume $K(x,y) \not\subseteq U$. Since $K(x,y)$, $U \in \overline{B}_x$ and $\overline{B}_x$ is a chain, we have $U \subseteq K(x,y)$. Hence $y \in K(x,y)$ which is contrary to (iv).

Let us define a function $d : M \times M \to \overline{B}_e$, $(x,y) \mapsto K(x,y)\varphi_x$. We claim that d is linear and $\mathcal{T} = \mathcal{T}_d$. By 3.6 (b) $\overline{B}_e$ is a chain.

It is enough for $T = T_d$ to show the conditions (1), (2) of 2.5. for (1): Let $x \in M$ and $\varepsilon \in \overline{B}_e \backslash \{\phi\}$. If x is not approximate, then there exists the smallest element U of $\mathcal{T}(x)$. Let $z \in U$. Then z is not seperated from x, hence by (iii) $d(x,z) = K(x,z)\varphi_x = \phi\varphi_x = \phi \subseteq \varepsilon$ i.e. $z \in K_\varepsilon(x)$.

Therefore $U \subseteq K_\varepsilon(x)$. If x is approximate, there exists for ε one $U \in B_x$ with $f_x(U) \subseteq \varepsilon$, since f_x is continuous at x and $\varepsilon \in \mathcal{T}(e)$. Now let $z \in U$. If z is not seperated from x then by (iii) $d(x,z) = K(x,z)\varphi_x = \phi\varphi_x = \phi \subseteq \varepsilon$. Hence let z be seperated from x. By (v) $K(x,z) \subseteq U$. With (i) $d(x,z) = K(x,z)\varphi_x \subseteq U\varphi_x$. We show next $U\varphi_x \subseteq \varepsilon$.

Let $V \in B_e$ with $f_x(U) \not\subseteq V$. Assume $V \not\subseteq \varepsilon$. Since $\varepsilon, V \in \overline{B}_e$ and $\overline{B}_e$ chain, $\varepsilon \subseteq V$, hence $f_x(U) \subseteq \varepsilon \subseteq V$. It is contrary to $f_x(U) \not\subseteq V$. Thus $V \subseteq \varepsilon$, i.e., $U\varphi_x \subseteq \varepsilon$ and with this $d(x,z) \subseteq \varepsilon$. For (2): Let $x \in M$ and $V \in \mathcal{T}(x)$. There exists then one $U \in B_x$ with $U \subseteq V$. By (ii) $U\varphi_x \in \overline{B}_e \backslash \{\phi\}$. Since e is approximate, ther is $\varepsilon \in \overline{B}_e \backslash \{\phi\}$ with $\varepsilon \subset U\varphi_x$. We show $K_\varepsilon(x) \subseteq U$. Let $z \in K_\varepsilon(x)$. If $z \notin U$, then $U \subseteq K(x,z)$ by definition of $K(x,z)$, hence $U\varphi_x \subseteq K(x,z)\varphi_x = d(x,z) \subseteq \varepsilon$. It is contrary to $\varepsilon \subset U\varphi_x$. It follows then $z \in U$, i.e., $K_\varepsilon(x) \subseteq U$. With $U \subseteq V$ $K_\varepsilon(x) \subseteq V$.

With 3.2, 3.4 and 3.7 we have obtained in the following the main theorem of this paragraph.

THEOREM 3.8. *A topological space is generated by a linear distance function if and only if the following conditions are satisfied.*

(1) Every $x \in M$ has a chain as a neighborhood basis.

(2) To every two approximate $x, y \in M$ there exists an x-isolation function $f : M \to M$ with $f(x) = y$.

Next we give a counterexample of a topological space which is not generated by a linear distance function.

DEFINITION 3.9. *A partial ordered set $(B, \leq)$ is called upper finite lattice if the following conditions are satisfied.*

(1) B has the largest element 1

(2) Every two elements of B has an infimum relative to $\leq$.

(3) For every $x \in B$ $[x, 1] := \{z | z \in B, x \leq z\}$ is finite.

THEOREM 3.10. *Let $(B, \leq)$ be an upper finite lattice. Then the following holds.*

(a) Every non-empty subset T of B has a supremum.

(b) Every chain in B relative to $\leq$ is at most countable.

Proof. (a): Let T be a non-empty subset of B. The set $S := \bigcap_{x \in T} [x, 1]$ is not empty because $1 \in S$. By 3.9 (3) S is finite.

Also by 3.9 (2) there exists the smallest element s of S which is obviously the supremum of T.

(b): Let K be a non-empty chain in B relative to $\leq$. We consider a function $K \to \mathrm{N}$, $x \mapsto |[x, 1]|$ and prove its injection. Let $x, y \in K$ with $|[x, 1]| = |[y, 1]|$. Without loss of generality we put $x \leq y$. Then $[y, 1] \subseteq [x, 1]$, i.e. $[x, 1] = [y, 1]$. Therefore $x = y$.

THEOREM 3.11. *Let $(B, \leq)$ be an upper finite lattice which itself is uncountable. For all $x \in M$ we denote $[\ , x] := \{z | z \in B, z \leq x\}$.*

Let $\mathcal{T} := \{T | T \subseteq B$, there exists one $b \in B$ such that $[\ , b] \subseteq T\} \cup \{\phi\}$. Then the following holds.

(a) $(B, \mathcal{T})$ is a topological space, in particular $\{1\} \notin \mathcal{T}$.

(b) $(B, \mathcal{T})$ will not be generated by a linear distance function.

Proof. (a) It is obviously enough to show : For all $S, T \in \mathcal{T}$, $S \cap T \in \mathcal{T}$. Let $S, T \in \mathcal{T}$ with $S, T \neq \phi$. There exists then $x, y \in B$ with $[\ , x] \subseteq S$, $[\ , y] \subseteq T$. Choosing $s := \inf\{x, y\}, [\ , s] \subseteq [\ , x] \cap [\ , y] \subseteq S \cap T$.

(b) Suppose there is a linear distance function d with $\mathcal{T} = \mathcal{T}_d$. By 3.2 the largest element 1 of B has a chain $\mathcal{A}$ as a neighborhood basis. For all $U \in \mathcal{A}$ $U \backslash \{1\} \neq \phi$ by (a). Now let $S := \{\sup(U \backslash \{1\}) | U \in \mathcal{A}\}$. Since $\mathcal{A}$ is a chain relative to $\subseteq$, S is a chain relative to $\leq$.

Hence by 3.10 (b) S itself is at most countable. Next it will be proved that to each $b \in B$ there is $s \in S$ with $s \leq b$. Let $b \in B$. If $b = 1$ then the assertion is trivial. Let hence $b \neq 1$. Then $\{1\} \cup [\ , b] \in \mathcal{T}(1)$. Since

$\mathcal{A}$ is a neighborhood basis of 1, there is $U \in \mathcal{A}$ with $U \subseteq \{1\} \cup [\ ,b]$. Hence $U \backslash \{1\} \subseteq [\ ,b]$ and therefore $s := \sup(U\backslash\{1\}) \leq b$. Because of the property which we have just proven, $\cup\{[s,1]|s \in S\} = B$. In other words B is a union of at most countable finite sets, i.e., is itself countable, which is contrary to the assumption.

EXAMPLE 3.12. *The set B of cofinite subsets of $\mathbf{R}$ with $\subseteq$ as a partial order is an upper finite lattice. By 3.11 we see that the linear distance function is not weak enough for the generation of every arbitrary topological space.*

4. Real Functions

DEFINITION 4.1. *Let M be a set. A linear distance function $d : M \times M \to \mathbf{R}^+ \cup \{0\}$ is said to be real.*

The next simple and well known theorem is useful to see the main theorem.

THEOREM 4.2. *Let $(M, \mathcal{T})$ be a topological space and $x \in M$. If x has a countable neighborhood basis, then there exists a descending sequence (V_n) of open neighborhoods of x relative to $\subseteq$ such that $V_1 := M$, so that $\{V_n | n \in \mathbf{N}\}$ is a neighborhood basis of x.*

THEOREM 4.3. *A topological space will be generated by a real distance function if and only if every point has countable neighborhood basis.*

Proof. Let $(M, \mathcal{T})$ be a topological space.
" $\longleftarrow$ ": Suppose every point $x \in M$ has a countable neighborhood basis. According to 4.2 every $x \in M$ has a descending sequence (V_n) of open sets with $V_1 = M$, so that $\{V_n | x \in \mathbf{N}\}$ is a neighborhood basis of x. We choose for every $x \in M$ such a sequence $(V_n(x))$ and define $d : M \times M \to \mathbf{R}^+ \cup \{0\}$, $(x,y) \mapsto \inf\{\frac{1}{n} \mid y \in V_n(x)\}$. We now claim that d is real and $\mathcal{T} = \mathcal{T}_d$. Before we prove it using 2.5, we show first: For all $m \in \mathbf{N}$ and $x \in M$ $K_{\frac{1}{m}}(x) = V_m(x)$.
Let $m \in \mathbf{N}$ and $x \in M$. The following statements are equivalent. $z \in K_{\frac{1}{m}}(x)$; $d(x,z) \leq \frac{1}{m}$; $\inf\{\frac{1}{n} | n \in \mathbf{N}, z \in V_n(x)\} \leq \frac{1}{m}$; there is $n_0 \in \mathbf{N}$

with $\frac{1}{n_0} \leq \frac{1}{m}$, $z \in V_{n_0}(x)$; there is $n_0 \in \mathbf{N}$ with $n_0 \geq m$, $z \in V_{n_0}(x)$; $z \in V_m(x)$ (since $(V_n(x))$ is a descending chain.).

Now for (1) of 2.5. Let $x \in M$ and $\varepsilon \in \mathbf{R}^+$. There is one $m \in \mathbf{N}$ with $\frac{1}{m} \leq \varepsilon$. Then $V_m(x) = K_{\frac{1}{m}}(x) \subseteq K_\varepsilon(x)$. For (2): Let $n \in \mathbf{N}$. Choose $\varepsilon := \frac{1}{n}$. Hence $K_\varepsilon(x) = K_{\frac{1}{n}}(x) = V_n(x)$. Therefore d is real with $\mathcal{T} = \mathcal{T}_d$.

" $\longrightarrow$ ": Let d be real with $\mathcal{T} = \mathcal{T}_d$. Since d is topological, for all $x \in M$ the set$\{K_{\frac{1}{n}}(x) | n \in \mathbf{N}\}$ is the countable neighborhood basis of x.

We construct here a topological space with which the equivalence of two functions can be clarified.

DEFINITION 4.4. *Under an ordinal number system of the type aleph 1 we understand a pair* $(\Omega, \underline{[})$ *where* Ω *is a set and* $\underline{[}$ *a relation on* Ω *with the following properties.*

(1) $\underline{[}$ *is a well order.*

(2) There exists the smallest element s *and the largest element* u *in* Ω.

(3) For all $x \in \Omega \backslash \{u\}$ $[s, x] := \{z | z \in \Omega,\ s \underline{[} z \underline{[} x\}$ *is countable.*

(4) Ω *itself is uncountable.*

For all $x \in \Omega \backslash \{u\}$ we denote $x' := \min\{z | z \in \Omega,\ x \underline{[} z\}$

THEOREM 4.5. *Let* $(\Omega, \underline{[})$ *be an ordinal number system of the type aleph 1. Let* $\mathcal{T} := \{[x, u] | x \in \Omega \backslash \{u\}\} \cup \{\phi\}$. *Then* $\mathcal{T}$ *is a topology on* Ω *and* $(\Omega, \mathcal{T})$ *will be generated by a linear distance function.*

Proof. Since $(\Omega, \underline{[})$ is a well ordered set, obviously $(\mathcal{T}, \supseteq)$ is also a well ordered set, hence it is a topology on Ω.

Using 3.8, we show that $(\Omega, \mathcal{T})$ is generated by a linear distance function. Since $\mathcal{T}$ itself is a chain, every $b \in \Omega$ has a chain as a neighborhood basis, namely $\mathcal{T}(b)$. Hence the condition (a) of 3.8 is satisfied. Obviously, for every $x \in \Omega \backslash \{u\}$ $[x, u]$ is the smallest open neighborhood of x. Thus every point of $\Omega \backslash \{u\}$ is not approximate and u is the only approximate point of Ω. Since for u, u we can choose the identity function as a u-isolation function, the condition of (b) is trivially satisfied.

THEOREM 4.6. *The topological space of 4.5 will not be generated by a real distance function.*

Proof. Every point $x \in \Omega\backslash\{u\}$ is seperated from u, because $x' \neq u, [x', u] \in \mathcal{T}(u)$ and $x \notin [x', u]$. Hence $\cap\mathcal{T}(u) = \{u\}$. Assume that $(\Omega, \mathcal{T})$ can be generated by a real distance function.

Then u has by 3.2 a countable neighborhood basis $\mathcal{A}$ and $\cap\mathcal{A} = \{u\}$. For every element $V \in \mathcal{A}$ $\Omega\backslash V$ is countable, because V contains one open set $[x, u]$ and from this $\Omega\backslash V \subseteq \Omega\backslash[x, u] \subseteq [s, x]$ where $[s, x]$ with $x \neq u$ is countable. Hence $\Omega\backslash\{u\} = \Omega\backslash(\cap\mathcal{A}) = \bigcup_{V\in\mathcal{A}}(\Omega\backslash V)$ is countable, which is contrary.

From 4.5 and 4.6 we conclude that the linear and the real distance functions are not equivalent. In other words the class of the topological spaces generated by a linear distance function does properly contain a class of those generated by a real distance function.

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