

## SOME REMARKS ON HOLOMORPHIC VECTOR BUNDLES OVER NON-KÄHLER MANIFOLDS

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If we consider an  $SU(2)$  vector bundle  $E$  over a Kähler surface  $S$ , then the moduli space of stable holomorphic structures on  $E$  is equal to the moduli space of anti-self-dual  $SU(2)$  connections on  $E$  if and only if  $b_1(S) = 0$ . This fact has a generalization for non-Kähler cases (2.4), (2.6), (2.7), (3.3). A modification of vanishing theorem is stated (1.10), which can be used to get a generalized Atiyah-Hitchin-Singer's elliptic complex on non-Kähler manifolds.

From now on our basic reference is [Kob]. Let  $M$  be a compact connected complex  $n$ -manifold with a hermitian metric  $g_{\mu\bar{\nu}}$  ( $1 \leq \mu, \nu \leq n$ ). The associated fundamental form will be denoted by

$$\Phi = \sqrt{-1} \sum g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu.$$

We do *not* assume that  $\Phi$  is a Kähler form, but we may and will assume that

$$(0.1) \quad d' d''(\Phi^{n-1}) = 0$$

after a conformal change of the metric, if necessary [Gau]. Such a metric will be called a *Gaudochon metric*.

### 1. Degree of bundles

For a holomorphic vector bundle  $\mathcal{E}$  over  $M$ , we define [Buc], [LY] the *degree* of  $\mathcal{E}$  relative to  $\Phi$  by

$$\deg(\mathcal{E}) = \deg_\Phi(\mathcal{E}) = \int_M c_1(\mathcal{E}, h) \wedge \Phi^{n-1} = \frac{1}{2n\pi} \int_M (\text{tr } K) \Phi^n,$$

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where  $c_1(\mathcal{E}, h)$  is the first Chern form associated to a hermitian metric  $h$  on  $\mathcal{E}$ ,  $\text{tr } K$  is the *scalar curvature* and  $K$  is the *mean curvature* [Kob]. The condition (0.1) implies that the degree is independent of the choice of  $h$ . Obviously  $\text{deg}(\mathcal{E}) = \text{deg}(\det \mathcal{E})$  and isomorphic bundles have the same degree. Thus we have a group homomorphism

$$\text{deg} : H^1(M, \mathcal{O}^\times) \rightarrow \mathbf{R}.$$

On Kähler manifolds degree is a topological invariant, but in non-Kähler case this is no longer true, i.e., there exists a hermitian manifold  $(M, \Phi)$  with a holomorphic line bundle  $\mathcal{L}$  such that  $c_1(\mathcal{L}) = 0 \in H^2(M; \mathbf{Z})$  and  $\text{deg } \mathcal{L} \neq 0$ . In particular,  $H^1(M, \mathcal{O}) \neq 0$  and the isomorphism class  $[\mathcal{L}]$  of  $\mathcal{L}$  generates an infinite cyclic subgroup in

$$\text{Pic}^0(M) = \{\ell \in H^1(M, \mathcal{O}^\times) \mid c_1(\ell) = 0 \in H^2(M; \mathbf{Z})\}.$$

For example, let  $\lambda$  be a nonzero complex number with  $|\lambda| \neq 1$ . Then on the Hopf manifold  $M = (\mathbf{C}^n - \{0\})/(z \mapsto \lambda z)$ , we consider the ‘metric’

$$\Phi = \frac{\sqrt{-1}}{|z|^2} (dz^1 \wedge d\bar{z}^1 + \dots + dz^n \wedge d\bar{z}^n),$$

which satisfies  $d' d''(\Phi^{n-1}) = 0$  (and  $d' d''(\Phi^{n-2}) \neq 0$  for  $n > 2$ . cf. (1.2)). Then it is easy to see that the mean curvature  $K$  of the *Chern connection* on the holomorphic tangent bundle  $\mathcal{T}$  of  $M$  is identically equal to  $n - 1$ . Thus  $M$  is an *Einstein-Hermitian manifold* and  $\text{deg } \mathcal{T} > 0$ . It follows (cf. (1.9)) that  $H^0(M, \Omega^p) = 0$  ( $1 \leq p \leq n$ ), where  $\Omega^p$  is the sheaf of holomorphic  $p$ -forms. Of course this can be obtained easily since there is no isolated singularity of a holomorphic function in  $\text{dim} > 1$ .

When  $n = 2$ , Buchdahl [Buc] found a necessary and sufficient condition for degree to be a topological invariant. In general we have the following. Let

$$\text{Pic}^0(M)_{\mathbf{R}} = \{\ell \in H^1(M, \mathcal{O}^\times) \mid c_1(\ell)_{\mathbf{R}} = 0 \in H^2(M; \mathbf{R})\}.$$

PROPOSITION 1.2. Consider the following statements.

- (1)  $b_1(M) = 2 \dim_{\mathbf{C}} H^1(M, \mathcal{O})$
- (2)  $\text{deg}(\text{Pic}^0(M)) = 0$
- (3)  $\text{deg}(\text{Pic}^0(M)_{\mathbf{R}}) = 0$
- (4) degree is a topological invariant.

Then (1) implies (2). (2), (3) and (4) are equivalent. If  $d'd''(\Phi^{n-2}) = 0$ , then (4) implies (1).

*Proof.* For the proof, we identify

$$(1.3) \quad \text{Pic}^0(M) \simeq H^1(M; \mathcal{O})/H^1(M; \mathbf{Z}) \simeq Z^{0,1}/B,$$

where

$$Z^{0,1} = \{\alpha \in A^{0,1}(M) \mid d''\alpha = 0\}$$

and

$$(1.4) \quad B = \{-d''g \cdot g^{-1} \mid g \in \mathcal{C}^\infty(M, \mathbf{C}^\times)\} \simeq \mathcal{C}^\infty(M, \mathbf{C}^\times)/\mathbf{C}^\times.$$

Note that  $B$  is a subgroup of  $Z^{0,1}$  containing

$$B^{0,1} = \{d''f \mid f \in \mathcal{C}^\infty(M, \mathbf{C})\}.$$

Also we have

$$(1.5) \quad B/B^{0,1} \simeq H^1(M; \mathbf{Z}).$$

Now the  $\deg |\text{Pic}^0(M)$  is defined by

$$(1.6) \quad \deg[\alpha] = \frac{\sqrt{-1}}{2\pi} \int_M (d'\alpha - d''\bar{\alpha}) \wedge \Phi^{n-1}$$

for  $[\alpha] \in \text{Pic}^0(M)$ ,  $\alpha \in Z^{0,1}$ .

Now suppose (1) is true. Then  $\text{Pic}^0(M)$  is a compact group and hence we get (2), which is obviously equivalent to (4).

Suppose (2) is true. Let  $\mathcal{L}$  be a holomorphic line bundle with  $c_1(\mathcal{L})_{\mathbf{R}} = 0 \in H^2(M; \mathbf{R})$ . Then for any hermitian metric  $h$  on  $\mathcal{L}$ ,  $c_1(\mathcal{L}, h)$  is a closed real (1,1)-form and hence there exists a  $\beta = \beta' + \beta'' \in A^{1,0} \oplus A^{0,1}$  such that  $c_1(\mathcal{L}, h) = \frac{\sqrt{-1}}{2\pi} d\beta$ . Then  $d''\beta = 0$ ,  $\beta' = -\bar{\beta}''$  and

$$c_1(\mathcal{L}, h) = \frac{\sqrt{-1}}{2\pi} (d'\beta'' - d''\bar{\beta}'').$$

Thus  $\deg(\mathcal{L}) = \deg[\beta''] = 0$ . This implies (3).

Obviously, (3) implies (2).

Finally, suppose  $d'd''(\Phi^{n-2}) = 0$  and (4) is true. By (1.5), for any  $\alpha \in Z^{0,1}$

$$\int_M d'\alpha \wedge \Phi^{n-1} = 0.$$

Then as in [Buc], there exists a unique  $\beta \in B^{0,1}$  such that

$$\Lambda d'(\alpha + \beta) = 0$$

for each  $\alpha \in Z^{0,1}$ . Then by the next observation, we have  $d'(\alpha + \beta) = 0$ .

**OBSERVATION.** Let  $\alpha \in Z^{0,1}$  and  $\Lambda d'\alpha = 0$ . Then  $d'\alpha = 0$  if  $d'd''(\Phi^{n-2}) = 0$ .

(For this observation, we do not need the assumption (0.1). This can be extended to “flat” holomorphic hermitian vector bundles.)

Now we obtain a map

$$\alpha \mapsto \overline{\alpha + \beta}$$

of  $Z^{0,1}$  into the space  $H^0(M, d\mathcal{O})$  of  $d$ -closed holomorphic 1-forms. This map induces an isomorphism

$$H^{0,1}(M) \simeq H^0(M, d\mathcal{O}).$$

This implies (1) [Kod].

**COROLLARY 1.7.** *On Kählerian manifolds, the degree relative to any Gauduchon metric is a topological invariant.*

**REMARK.** *The condition  $d'd''(\Phi^{n-2}) = 0$  implies that, for instance,*

$$\int_M c_2(\mathcal{E}, h) \wedge \Phi^{n-2}$$

*is independent of the choice of  $h$  [BC]. Hence one can obtain Lübke inequality [L1] and the lower bound for the Yang-Mills functional.*

Next proposition is trivial.

**PROPOSITION 1.8.** *If degree is a topological invariant on  $M$  and  $b_2(M) = 0$ , then there are no stable bundles of  $\text{rk} > 1$ . Every holomorphic vector bundle is semi-stable and every Einstein-Hermitian vector bundle is a direct sum of line bundles with the same degree.*

The following vanishing theorem indicates a role of degree.

**VANISHING THEOREM 1.9 [Kob].** *Let  $(\mathcal{E}, h)$  be an Einstein-Hermitian vector bundle over a Hermitian manifold  $(M, \Phi)$ . If  $\text{deg}(\mathcal{E}) < 0$ , then  $\mathcal{E}$  has no holomorphic section. If  $\text{deg}(\mathcal{E}) = 0$ , then every section of  $\mathcal{E}$  is parallel.*

Since every holomorphic line bundle admits an Einstein-Hermitian metric, the vanishing theorem applies to any holomorphic line bundle. This vanishing theorem has a following generalization.

**PROPOSITION 1.10.** *Let  $(\mathcal{E}, h)$  be a hermitian holomorphic vector bundle over  $(M, \Phi)$ . Let  $D = D' + D''$  be the Chern connection on  $(\mathcal{E}, h)$  and  $u$  be a smooth section of  $\mathcal{E}$ .*

- (1) *If  $K \leq 0$  and  $\Lambda D' D'' u = 0$ , then  $Du = 0$ . If, moreover,  $K < 0$  at some point of  $M$ , then  $u = 0$ .*
- (2) *If  $K \geq 0$  and  $\Lambda D'' D' u = 0$ , then  $Du = 0$ . If, moreover,  $K > 0$  at some point of  $M$ , then  $u = 0$ .*

*Proof.* Observe that if  $\Lambda D' D'' u = 0$ ,

$$\sqrt{-1} \Lambda d' d'' h(u, u) = |Du|^2 - h(Ku, u).$$

Then the maximum principle of E. Hopf applies. (2) is similarly proved.

This vanishing theorem can be used to get a generalized Atiyah-Hitchin-Singer's elliptic complex (cf. [AHS], [K2]) for an Einstein-Hermitian connection on a hermitian manifold.

## 2. Holomorphic structures

Now we fix a smooth complex vector bundle  $E$  over  $M$  of rank  $r$ . There are three important concepts on  $E$ , namely, holomorphic structures, unitary structures and connections. The sets of these structures

will be denoted by  $\text{Hol}(E)$ ,  $\text{Herm}(E)$  and  $\text{Con}(E)$ , respectively. Then there is a *Chern map*

$$\text{Hol}(E) \times \text{Herm}(E) \rightarrow \text{Con}(E).$$

The group  $\text{GL}(E)$  of smooth bundle automorphisms of  $E$  acts naturally on these spaces and the Chern map is equivariant. The Chern map is *natural* in the sense that for any vector bundle  $\rho(E)$  associated to  $E$ , the diagram

$$\begin{array}{ccc} \text{Hol}(E) \times \text{Herm}(E) & \longrightarrow & \text{Con}(E) \\ \downarrow & & \downarrow \\ \text{Hol}(\rho(E)) \times \text{Herm}(\rho(E)) & \longrightarrow & \text{Con}(\rho(E)) \end{array}$$

commutes equivariantly. We consider only the case  $\rho(E) = \det E$ , since we have a complete understanding in that situation. A different point of view is considered in [New], [OV], [L2].

From now on we will assume that  $\text{Hol}(E) \neq \emptyset$ . Then there is a commutative diagram

$$(2.1) \quad \begin{array}{ccc} \text{Hol}(E) & \xrightarrow{\det} & \text{Hol}(\det E) \\ \downarrow & & \downarrow \\ \mathcal{M}(E) & \longrightarrow & \mathcal{M}(\det E) \end{array}$$

where  $\mathcal{M}(E) = \text{Hol}(E)/\text{GL}(E)$ , which we may call the *moduli space of holomorphic structures* on  $E$ . We identify ([Gri], [AHS], [AB], [Qui], [Kob], [K2]) a holomorphic structure with the corresponding *Cauchy-Riemann operator*  $D'' : A^0(E) \rightarrow A^{0,1}(E)$ ,  $D'' \circ D'' = 0$ . They form a subset of an affine space, of which the model space is  $A^{0,1}(\text{End } E)$ . Thus  $\text{Hol}(E)$  and hence  $\mathcal{M}(E)$  is canonically equipped with a smooth topology [Pal]. Note that there is a simple transitive action of the group  $\text{Pic}^0(M)$  on  $\mathcal{M}(\det E)$  and hence  $\mathcal{M}(\det E)$  is (noncanonically) isomorphic to  $\text{Pic}^0(M)$ . The *surjective map*

$$(2.2) \quad \det : \text{Hol}(E) \rightarrow \text{Hol}(\det E)$$

is a trivial fiber bundle. Once a holomorphic structure or equivalently a Cauchy-Riemann operator  $D''$  is chosen, a trivialization of  $\text{Hol}(E)$  over  $\text{Hol}(\det E)$  is given by

$$\text{Hol}(E) \simeq \text{Hol}(\det E) \times \{\beta \in A^{0,1}(\text{End } E) : \text{tr } \beta = 0, D''(\beta) + \beta \circ \beta = 0\}.$$

The fiber of (2.2) at  $\mathcal{L} \in \text{Hol}(\det E)$  is denoted by

$$\text{Hol}(E, \mathcal{L}) = \{\mathcal{E} \in \text{Hol}(E) \mid \det \mathcal{E} = \mathcal{L}\},$$

and

$$\mathcal{M}(E, \mathcal{L}) := \text{Hol}(E, \mathcal{L}) / \text{SL}(E),$$

where

$$\text{SL}(E) = \{g \in \text{GL}(E) \mid \det g = 1\}.$$

The fiber bundle

$$(2.3) \quad \mathcal{M}(E) \rightarrow \mathcal{M}(\det E)$$

becomes trivial after it is divided by a finite group (2.4). The group  $\text{Pic}^0(M)$  also acts on  $\mathcal{M}(E)$ , by tensoring, and the induced action on  $\mathcal{M}(E)$  of the  $r$ -torsion subgroup

$$T := T_r = \{\ell \in \text{Pic}^0(\tilde{M}) \mid r\ell = 0\}$$

commutes with the projection  $\mathcal{M}(E) \rightarrow \mathcal{M}(\det E)$ . Note that  $T$  is a finite group isomorphic to  $(\mathbf{Z}/r\mathbf{Z})^{b_1}$ , where  $b_1$  is the first Betti number of  $M$ . Although the stabilizers in  $T$  are not simply described, we have

**PROPOSITION 2.4.**  $\mathcal{M}(E)/T$  is isomorphic to the product  $\mathcal{M}(\det E) \times (\mathcal{M}(E)/\text{Pic}^0(M))$  as spaces over  $\mathcal{M}(\det E)$ .

*Proof.* Probably, the proof using the Cauchy-Riemann operators might be more clear. But here is the direct proof. The isomorphism  $\mathcal{M}(E)/T \rightarrow \mathcal{M}(\det E) \times (\mathcal{M}(E)/\text{Pic}^0(M))$  is given by

$$[\mathcal{E}]_T \mapsto [\det \mathcal{E}] \times [\mathcal{E}]_{\text{Pic}^0(M)}.$$

Obviously this is a well-defined continuous map. To see the injectivity, suppose

$$[\det \mathcal{E}_1] \times [\mathcal{E}_1]_{\text{Pic}^0(M)} = [\det \mathcal{E}_2] \times [\mathcal{E}_2]_{\text{Pic}^0(M)}.$$

Then  $[\det \mathcal{E}_1] = [\det \mathcal{E}_2]$  and  $[\mathcal{E}_1]_{\text{Pic}^0(M)} = [\mathcal{E}_2]_{\text{Pic}^0(M)}$ . Thus there exists a  $[\mathcal{L}] \in \text{Pic}^0(M)$  such that  $\mathcal{E}_1 \otimes \mathcal{L} \simeq \mathcal{E}_2$ . Then  $\det \mathcal{E}_1 \otimes \mathcal{L}^r \simeq \det \mathcal{E}_2$ . Thus  $\mathcal{L}^r \simeq \mathcal{O}$ , i.e.,  $[\mathcal{L}] \in T$ . Hence  $[\mathcal{E}_1]_T = [\mathcal{E}_2]_T$ .

For the surjectivity, let  $[\mathcal{L}] \times [\mathcal{E}_1] \in \mathcal{M}(\det E) \times \mathcal{M}(E)$  be given. Then

$$[\mathcal{L}] = [\det \mathcal{E}_1] + \ell$$

for some unique  $\ell \in \text{Pic}^0(M)$ . Since  $\text{Pic}^0(M)$  is a divisible group, there exists a  $\ell_1$  such that  $\ell = r\ell_1$ . Locally, this  $\ell_1$  can be chosen continuously. Then we put  $[\mathcal{E}] = [\mathcal{E}_1] \otimes \ell_1$ . Then  $[\mathcal{E}]_T \in \mathcal{M}(E)/T$  is independent of the choice of  $\ell_1$  and  $[\mathcal{E}]_T$  maps to  $[\mathcal{L}] \times [\mathcal{E}_1]_{\text{Pic}^0(M)}$ . This establishes the isomorphism.

**LEMMA 2.5.** *The followings are equivalent.*

- (1)  $b_1(M) = 0$
- (2)  $\mathcal{C}^\infty(M, \mathbf{C}^\times)$  is a divisible group
- (3)  $\mathcal{C}^\infty(M, \mathbf{C}^\times)$  is connected.
- (4)  $\mathcal{C}^\infty(M, \mathbf{C}^\times)/\mathbf{C}^\times$  is a divisible group
- (5)  $\mathcal{C}^\infty(M, \mathbf{C}^\times)/\mathbf{C}^\times$  is connected.
- (6)  $\text{Pic}^0(M)$  has no torsion
- (7)  $\text{Pic}^0(M) \simeq H^1(M, \mathcal{O})$ .

Moreover these imply that  $\text{Pic}^0(M)$  acts freely on  $\mathcal{M}(E)$ .

Now we get (cf. [K3], [OV] [L2])

**COROLLARY 2.6.** (1) *If  $b_1 = 0$ , then  $\mathcal{M}(E, \mathcal{L}) \simeq \mathcal{M}(E)/\text{Pic}^0(M)$  for any  $\mathcal{L} \in \text{Hol}(\det E)$ .*

(2) *If  $H^1(M, \mathcal{O}) = 0$ , then  $\mathcal{M}(E, \mathcal{L}) \simeq \mathcal{M}(E)$  for any  $\mathcal{L} \in \text{Hol}(\det E)$ .*

*Proof.* (1) Since  $b_1 = 0$ ,  $T = 0$  and hence by (2.4)  $\mathcal{M}(E) \simeq \mathcal{M}(\det E) \times (\mathcal{M}(E)/\text{Pic}^0(M))$  as spaces over  $\mathcal{M}(\det E)$ . In particular, the fiber  $\mathcal{M}(E)_{[\mathcal{L}]}$  of  $\mathcal{M}(E) \rightarrow \mathcal{M}(\det E)$  at  $[\mathcal{L}] \in \mathcal{M}(\det E)$  is isomorphic to



$\mathcal{M}(E)/\text{Pic}^0(M)$ . Thus it suffices to show that  $\mathcal{M}(E)_{[\mathcal{L}]} \simeq \mathcal{M}(E, \mathcal{L})$ . From the commutative diagram (2.1), we have an injection

$$\mathcal{M}(E, \mathcal{L}) \rightarrow \mathcal{M}(E)_{[\mathcal{L}]}.$$

To see the surjectivity of this map, let  $[D''] \in \mathcal{M}(E)$  and  $[\det D''] = [\mathcal{L}]$  (i.e.,  $[D''] \in \mathcal{M}(E)_{[\mathcal{L}]}$ ). Then  $\det D'' = \mathcal{L} - \beta_1$  for some  $\beta_1 \in B \simeq C^\infty(M, C^\times)/C^\times$  (cf. (1.4)). Since  $B$  is divisible,  $\beta_1 = r\beta$  for some (unique)  $\beta \in B$ . Now

$$[D'' + \beta 1_E] = [D'']$$

and  $\det(D'' + \beta 1_E) = \det D'' + \text{tr}(\beta 1_E) = \mathcal{L}$ . This establishes a homeomorphism.

(2) follows from (1).

REMARKS 2.7. (1) If we consider stable structures ( $[\mathbf{Buc}]$ ,  $[\mathbf{LY}]$ ), then we have propositions similar to (2.4) and (2.6) with  $\mathcal{M}^s(E) := \text{Hol}^s(E)/\text{GL}(E)$  and  $\mathcal{M}^s(E, \mathcal{L}) := \text{Hol}^s(E, \mathcal{L})/\text{SL}(E)$ .

(2) If the map  $\mathcal{M}(E, \mathcal{L}) \hookrightarrow \mathcal{M}(E)$  is surjective, then obviously  $\text{Pic}^0(M) = 0$ , i.e.,  $H^1(M, \mathcal{O}) = 0$ .

### 3. Einstein-Hermitian connections

From now on we fix a unitary structure  $h$  on  $E$ . The space of irreducible Einstein  $h$ -connections on  $E$  is denoted by  $\mathcal{C}^s(E)$  and

$$\mathcal{N}^s(E) := \mathcal{C}^s(E)/\text{U}(E),$$

where  $\text{U}(E)$  is the group of smooth isometries of  $(E, h)$ . We assume that  $\mathcal{C}^s(E) \neq \emptyset$ . Then as in the previous section we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{C}^s(E) & \xrightarrow{\det} & \mathcal{C}^s(\det E) \\ \downarrow & & \downarrow \\ \mathcal{N}^s(E) & \longrightarrow & \mathcal{N}^s(\det E) \end{array}$$

The map  $\det : \mathcal{C}^s(E) \rightarrow \mathcal{C}^s(\det E)$  is a trivial fibration. Once a point  $D \in \mathcal{C}^s(E)$  is chosen, the trivialization is given by

$$\mathcal{C}^s(E) \simeq \mathcal{C}^s(\det E) \times \{A \in A^1(uE) \mid D''(A'') + A'' \circ A'' = 0, \\ \Lambda(D(A) + A \circ A) = 0\},$$

where  $uE$  is the real vector bundle of skew-hermitian endomorphisms of  $(E, h)$ . We put for  $\nabla \in \mathcal{C}^s(\det E)$ ,

$$\mathcal{C}^s(E, \nabla) = \{D \in \mathcal{C}^s(E) \mid \det D = \nabla\}$$

and

$$\mathcal{N}^s(E, \nabla) = \mathcal{C}^s(E, \nabla) / \text{SU}(E),$$

where  $\text{SU}(E) = \text{U}(E) \cap \text{SL}(E)$ . Then

(3.1)  $\mathcal{N}^s(E) \simeq \mathcal{M}^s(E)$  ([LY]) and hence  $\mathcal{M}^s(E)$  is an open subset of  $\mathcal{M}(E)$  ([K1], cf. [Kob]).

(3.2) Let  $\mathcal{L}$  be the holomorphic structure on  $\det E$  defined by  $\nabla \in \mathcal{C}^s(\det E)$ . Then  $\mathcal{N}^s(E, \nabla) \simeq \mathcal{M}^s(E, \mathcal{L})$  and hence  $\mathcal{M}^s(E, \mathcal{L})$  is an open subset of  $\mathcal{M}(E, \mathcal{L})$ .

(3.3) When  $r = n = 2$ ,  $\mathcal{M}^s(E)$  is the ordinary moduli space  $\mathcal{M}(c_1, c_2)$  considered in algebraic geometry and  $\mathcal{N}^s(E, d)$  is, if  $c_1(E) = 0$  and  $\nabla = d$ , the moduli space of anti-self-dual  $SU(2)$ -connections. These two spaces are equal if and only if  $H^1(M, \mathcal{O}) = 0$  (cf. (2.6) and (2.7)).

(3.4) If

$$\mathcal{M}_*^s(E) = \{[\mathcal{E}] \in \mathcal{M}^s(E) \mid H^2(M, sl\mathcal{E}) = 0\}$$

and

$$\mathcal{M}_*^s(E, \mathcal{L}) = \{[\mathcal{E}] \in \mathcal{M}^s(E, \mathcal{L}) \mid H^2(M, sl\mathcal{E}) = 0\}$$

then  $\mathcal{M}_*^s(E)$  (resp.  $\mathcal{M}_*^s(E, \mathcal{L})$ ) is a Kähler manifold and the tangent space at  $[\mathcal{E}]$  is isomorphic to  $H^1(M, \text{End } \mathcal{E})$  (resp.  $H^1(M, sl\mathcal{E})$ ) (cf. [K2], [Kob]), where  $sl\mathcal{E}$  is the bundle of trace-free endomorphisms of  $\mathcal{E}$ .

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