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IRREDUCIBILITY OF THE HILBERT SCHEME OF SMOOTH COMPLEX SPACE CURVES OF LOW DEGREE

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In this paper, we show that the Hilbert scheme $\mathcal{I}'_{d,g,3}$ of smooth curves of degree d in $\mathbf{P}^3$ is irreducible in case i) $d = 11$, $g = 10$ and ii) $d = 10$, $g = 9$. Together with the results in [9], one can then see that $\mathcal{I}'_{d,g,3}$ is irreducible for all d and g such that $d > g$ with positive Brill-Noether number $\rho(d, g, 3) = g - 4(g - d + 3)$.

In proving the main theorem of this paper, we utilize some of the recent results of Coppens [5] about the variety of special linear systems on a fixed algebraic curve as well as similar techniques used in [9]. Throughout we will be working over the field of complex numbers.

Basic set up, terminologies and preliminary results

First recall that, given non-negative integers r, d, for every point p of the moduli space $\mathcal{M}_g$ of smooth curves of genus g and any sufficiently small connected neighborhood U of p, there are a smooth connected variety $\mathcal{M}$, a finite ramified covering :

$$h : \mathcal{M} \to U$$

and two varieties, proper over $\mathcal{M}$:

$$\xi : \mathcal{C} \to \mathcal{M}, \ \pi : \mathcal{G}^r_d \to \mathcal{M}$$

with the following properties:

1) $\mathcal{C}$ is a universal curve over $\mathcal{M}$, i.e. for every $p \in \mathcal{M}$, $\xi^{-1}(p)$ is a smooth curve of genus g whose isomorphism class is $h(p)$.

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2) $\mathcal{G}_d^r$ parametrizes pairs $(p, \mathcal{D})$, where $p \in \mathcal{M}$ and $\mathcal{D}$ is a linear system (possibly incomplete) of degree d and of dimension r, which is denoted by g_d^r, on $C = \xi^{-1}(p)$.

Let $\mathcal{G}$ be the union of irreducible components of $\mathcal{G}_d^r$ whose general element correspond to pairs $(p, \mathcal{D})$ such that $\mathcal{D}$ is a very ample linear system on $\xi^{-1}(p) = C$, i.e. $\mathcal{D}$ gives an embedding of C into $\mathbf{P}^r$. We will prove that $\mathcal{G}$ is irreducible for i) $d = 11,\ g = 10$ and ii) $d = 10,\ g = 9$. And this will in turn guarantee the irreducibility of the scheme $\mathcal{I}'_{d,g,3}$ for the cases. One of the several key facts we will utilize for the proof of our main assertion is the following, which has been observed and used in several circumstances: see [1] or [6].

PROPOSITION 1.1. *There exists a unique component $\mathcal{G}_0$ of $\mathcal{G}$ which dominates $\mathcal{M}$ (or $\mathcal{M}_g$) if the Brill-Noether number*

$$\rho(d, g, r) = g - (r+1)(g - d + r)$$

is positive.

REMARK 1.2. *In the Brill-Noether range, i.e. in the range $\rho(d, g, r) > 0$, we call the component $\mathcal{G}_0$ of $\mathcal{G}$ which dominates $\mathcal{M}$ the principal component.*

The following facts are also useful for the proof of our main assertion whose proofs can be found in [2].

PROPOSITION 1.3. *(i) Any component of $\mathcal{G}_d^r$ has its dimension at least $3g - 3 + \rho(d, g, r)$.*

(ii) Suppose $g > 0$ and let X be a component of $\mathcal{G}_d^2$ whose general element $(p, \mathcal{D})$ is such that $\mathcal{D}$ is a linear system on $C = \xi^{-1}(p)$ which is not composed with an involution. Then

$$\dim X = 3g - 3 + \rho(d, g, 2) = 3d + g - 9.$$

(iii) For any d, $\mathcal{G}_d^1$ is smooth of dimension $2d + 2g - 5$, if $g > 1$.

COROLLARY 1.4. *Let $\mathcal{W}$ be an irreducible closed subvariety of $\mathcal{G}_d^r$, $r \geq 2$, whose general member $(p, \mathcal{D})$ is such that $\mathcal{D}$ is complete, special and* *birationally very ample on* $C = \xi^{-1}(p)$. *Then*

$$\dim \mathcal{W} \leq 3d + g - 4r - 1.$$

Proof. See [8].

Irreducibility of $\mathcal{I}'_{11,10,3}$ and $\mathcal{I}'_{10,9,3}$.

With these preparations, we now prove the irreducibility of $\mathcal{I}'_{d,g,3}$ for $d = g+1$ with $g = 10$ and 9, which have been left open in [9]. But before proceeding, it should be noted that we are primarily interested in the irreducibility of $\mathcal{I}'_{d,g,3}$ only for those d and g in the Brill-Noether range, so that we can use Proposiiton (1.1) for the investigation of the matter. On the other hand, if $d > 2g-2$ it is quite elementary to see that $\mathcal{I}'_{d,g,r}$ is irreducible if $g - d + r \leq 0$ and is empty otherwise. Thus for the case $r = 3$ we may well assume that $\frac{3}{4}g + 3 < d \leq 2g - 2$, and hence $g \geq 9$ in case $d = g+1$. And the irreducibility of $\mathcal{I}'_{d,g,r}$ for $d = g+1$ with $g \geq 11$ has been proven in [9].

REMARK 2.1. *i) Given a fixed smooth curve* C, *we denote the variety of* g^r_d*'s on* C *by* $G^r_d(C)$. *The description of the tangent space to* $G^r_d(C)$ *at a point* $\mathcal{D} = \mathbf{P}(V)$ *where* $V \subseteq H^0(C, L)$, $L = \mathcal{O}(D)$, $D \in \mathcal{D}$ *is well known: see [2] for details. In particular we have*

$$\dim T_{\mathcal{D}}(G^r_d(C)) = \rho(d, g, r) + \dim Ker\, \mu_0$$

where μ_0 *is the cup product map*

$$\mu_0 : V \otimes H^0(C, K_C L^{-1}) \to H^0(C, K).$$

ii) Denoting by $W^r_d(C)$ *the variety of complete linear systems of degree* d *and of dimension at least* r *on a fixed algebraic curve* C, *the following results of M. Coppens as well as Lemma (2.5) play an important role for the proof of the main result of this paper.*

LEMMA 2.2. *(Coppens) Let* C *be an algebraic curve of genus* g. *Then the following hold.*
i) If $g = 10$ *and* $\dim W^1_7(C) = 3$ *then* $\dim W^1_6(C) = 2$.

ii) If $g = 9$ *and* $\dim W_6^1(C) = 2$ *then* $\dim W_5^1(C) = 1$.

Proof. See [5], Proposition 12.

LEMMA 2.3. *Let* C *be an algebraic curve of genus* g. *Then the following hold.*
(i) If $g = 10$ *and* $\dim W_6^1(C) = 2$ *then* C *has finitely many* g_4^1*'s unless* C *is a smooth plane sextic.*
(ii) If $g = 9$ *and* $\dim W_5^1(C) = 1$ *then* C *has finitely many* g_4^1*'s.*

Proof. See [7], Theorem (3.3) or [3], p.200. See also [5], Proposition 12.

LEMMA 2.4. *Let* $d = g + 1$, $r = 3$, $g = 9$ *or* $g = 10$. *Suppose that* $\mathcal{G}'$ *is a component of* $\mathcal{G}$ *other than* $\mathcal{G}_0$. *Then a general element* $(p, \mathcal{D})$ *of* $\mathcal{G}'$ *is such that* $\mathcal{D}$ *is an incomplete linear system on* $C = \xi^{-1}(p)$.

Proof. Suppose that a general element $(p, \mathcal{D})$ of $\mathcal{G}'$ corresponds to a complete $\mathcal{D}$ on $C = \xi^{-1}(p)$. Let f be the degree of the fixed divisor of the residual linear system $|K - D|$ where $D \in \mathcal{D}$. Then we have the natural map

$$\Psi : \mathcal{V} \to \mathcal{G}^1_{g-3-f}$$

defined by $\Psi(p, \mathcal{D}) = (p, |K - D| - F)$, where $\mathcal{V}$ is an open subset of $\mathcal{G}'$, $D \in \mathcal{D}$ and F is the fixed divisor of the linear system $|K - D|$. Then by Proposition (1.3) we must have,

$$\begin{aligned} 3g - 3 + \rho(g+1, g, 3) \leq \dim \mathcal{G}' = \dim \mathcal{V} &\leq \dim \mathcal{G}^1_{g-3-f} + f \\ &= 3g - 3 + \rho(g - 3 - f, g, 1) + f \end{aligned}$$

which implies $f = 0$. Thus we deduce that for a general element $(p, |D|)$ of $\mathcal{G}'$, the residual linear system $|K - D| = |E|$ is a base-point-free pencil. Since $\mathcal{G}'$ does not dominate $\mathcal{M}$, $G_d^3(C)$ has dimension more than the Brill-Noether number $\rho(g + 1, g, 3)$. Thus by considering the cup product map

$$\mu_0 : H^0(C, \mathcal{O}(D)) \otimes H^0(C, K_C \otimes \mathcal{O}(-D)) \to H^0(C, K_C),$$

(*)
$$\rho(g+1, g, 3) < \dim G_d^3(C) \leq \dim T_{|D|} G_d^3(C) = \rho(g+1, g, 3) + \dim Ker\, \mu_0$$

and we see that dim $Ker\,\mu_0 > 0$. On the other hand, by the base-point-free-pencil trick, $Ker\,\mu_0 \cong H^0(C, K_C \otimes \mathcal{O}(-2E))$. Note that $\deg|K-2E| = 4$ and by Clifford theorem $r(K-2E) \leq 1$ since C cannot be a hyperelliptic curve. We now consider the following two possibilities

(i) If $r(K-2E) = 1$, then we have the natural map

$$\Psi' : \mathcal{V}' \to \mathcal{G}_4^1$$

defined by $\Psi'(p, |D|) = |K-2E|$, where $\mathcal{V}'$ is an open subset of $\mathcal{G}'$, and $|E| = |K-D|$. The map Ψ' is easily seen to be a finite to one map and hence

$$\dim \mathcal{V}' = \dim \mathcal{G}' \leq \dim \mathcal{G}_4^1 = 2g+3 < 3g-3+\rho(g+1,g,3) = 4(g+1)-15,$$

which is contradictory to Proposition (1.3), (i).

(ii) If $r(K-2E) = 0$, then by the inequality (*) and by the hypothesis that a general element $(p, \mathcal{D})$ of $\mathcal{G}'$ corresponds to a complete $\mathcal{D}$ on $C = \xi^{-1}(p)$,

$$\dim G_{g+1}^3(C) = \dim W_{g+1}^3(C) = g-7.$$

Then by passing to residual linear system, we have $\dim W_{g-3}^1(C) = g-7$:

(a) If $g = 10$, then $\dim W_6^1(C) = 2$, by Lemma (2.2), (i). Thus C is either a smooth plane sextic or a curve with a base-point-free g_4^1 by Lemma (2.3),(i). Since the residual linear system $|E| = |K-D| \in W_6^1(C)$ of a general element $(p, |D|)$ of $\mathcal{G}'$ is a base-point-free pencil and since a smooth plane sextic cannot have a base-point-free g_6^1, C cannot be a smooth plane sextic.

(b) If $g = 9$, then $\dim W_5^1 = 1$, by Lemma (2.2),(ii). Thus C is a curve with a base-point-free g_4^1 by Lemma (2.3),(ii).

Accordingly we have the projection map

$$\pi : \mathcal{V}'' \to \mathcal{M}_4^1 \subset \mathcal{M}$$

where $\mathcal{V}''$ is an open subset of $\mathcal{G}'$ and $\mathcal{M}_4^1$ is the sublocus in $\mathcal{M}$ consisting points $p \in \mathcal{M}$ such that $\xi^{-1}(p) = C$ has a g_4^1. Since the dimension of the fiber of the map π over a point in the image is $g-7$, we have

$$\dim \mathcal{G}' = \dim \mathcal{V}'' \leq \dim \mathcal{M}_4^1 + (g-7) = (2g+3)+(g-7) < 4(g+1)-15,$$

which is contradictory to Proposition (1.3),(i).

LEMMA 2.5. *Let C be a smooth curve of genus g. Let G be an irreducible subvariety of $G^r_d(C)$. Let $\mathcal{D}$ be a general point of G. Suppose $|\mathcal{D}|$ is birationally very ample. Let D be a divisor belonging to $\mathcal{D}$. If D is special,*

$$\dim G \leq 2d - g - 3r + 1 + h^0(C, K_C \otimes \mathcal{O}(-2D)).$$

Proof. See [AC-1] or [6].

THEOREM 2.6. *(i) $\mathcal{G}$ is irreducible if $d = g+1$ where $g = 9$ or 10.*
(ii) $\mathcal{I}'_{11,10,3}$ and $\mathcal{I}'_{10,9,3}$ is irreducible.

Proof. We only need to prove (i). Let $\mathcal{G}'$ be a component of $\mathcal{G}$ other than the principal component $\mathcal{G}_0$. Let $(p, \mathcal{D})$ be a general element of $\mathcal{G}'$ and $r = r(D)$, $D \in \mathcal{D}$. By Lemma (2.4) we have $r \geq 4$. Consider the morphism

$$\zeta : \mathcal{V} \to \mathcal{G}^r_d$$

defined by $\zeta(p, \mathcal{D}) = (p, |D|)$ where $\mathcal{V}$ is an open subset of $\mathcal{G}'$ and $D \in \mathcal{D}$. Let $\mathcal{W}$ be the image of the morphism ζ. Let $(p, |D|)$ be a general element of $\mathcal{W}$ and f be the degree of the fixed divisor F of the linear system $|E| = |K - D|$. By passing to residual series and then taking off the fixed points, we have the map

$$\Phi : \mathcal{V} \xrightarrow{\zeta} \mathcal{W} \xrightarrow{\lambda} \mathcal{G}^{r-2}_{g-1-f}.$$

We now consider the following possibilities.
(i) $|E - F|$ is birationally very ample:

By applying Corollary (1.4) and by noting the fact that the dimension of the fiber of the morphism ζ over a general point in $\mathcal{W}$ is $4(r-3)$, we have

$$\dim \mathcal{G}' = \dim \mathcal{V} \leq 3(g-1-f)+g-4(r-2)-1+f+4(r-3) < 4(g+1)-15,$$

which is contradictory to Proposition (1.3),(i).
(ii) $|E - F|$ is not birationally very ample:

Denoting by $\chi_{n,\gamma}$ the closure of the subvariety of $\mathcal{M}$ consisting of all points corresponding to curves which are n-fold ramified coverings of smooth genus γ curves, we note, by Riemann's moduli count,

$$\dim \chi_{n,\gamma} \leq 2g + (2n-3)(1-\gamma) - 2.$$

Let n and γ be the degree of the map from C determined by the linear system $|E-F|$ and the genus of the image curve respectively.

Suppose that $\gamma \geq 1$: By applying Lemma (2.5) to a general fiber of the map

$$\mathcal{W} \xrightarrow{\lambda} \mathcal{G}^{r-2}_{g-1-f} \xrightarrow{\pi} \chi_{n,\gamma}$$

we have

$$\dim \mathcal{W} \leq \dim \chi_{n,\gamma} + 2(g+1) - g - 3r + 1 \leq (2g-2) + 2(g+1) - g - 3r + 1.$$

Since $|D|$ is birationally very ample, $r(2D) \geq 3r-1$ and hence $r \leq \frac{g+3}{3}$. Thus one gets

$$\dim \mathcal{V} = \dim \mathcal{W} + 4(r-3) \leq 3g + r - 11 < 4(g+1) - 15,$$

which is contradictory to Proposition (1.3),(i).

Suppose that $\gamma = 0$: In this case we have $|E| = (i-1)g^1_n + F$, $f = \deg F = (g-3) - (i-1)n$, where $i = h^0(C, K_C \otimes \mathcal{O}(-D)) = r-1$. We also note that $n \geq 3$ since a hyperelliptic curve cannot have a birationally very ample special linear system $\mathcal{D}$. Thus we have

$$\begin{aligned}\dim \mathcal{V} = \dim \mathcal{W} + 4(r-3) &\leq \dim \mathcal{G}^1_n + f + 4(r-3)\\ &= 3(g+1) - (n-4)(r-4) - 7\\ &\leq 3(g+1) + (r-4) - 7 = 3g + r - 8.\end{aligned}$$

On the other hand, since $r \leq \frac{g+3}{3}$ we have

$$\dim \mathcal{V} \leq 3g + r - 8 < 4(g+1) - 15,$$

which is contradictory to Proposition (1.3),(i).

Finally, one immediately gets the following improvement of the main result of [9].

THEOREM 2.7. *$\mathcal{I}'_{d,g,3}$ is irreducible for all d and g such that $d > g$ with positive Brill-Noether number $\rho(d,g,3) = g - 4(g - d + 3)$.*

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