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ON f-V-RINGS

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A ring R is called a *left V-ring* if every simple left R-module is injective(C. Faith [1]), and its connections with von Neumann regular rings are extensively studied (cf. [2]). In [8,9], R is called a *left f-V-ring (resp., f-GV-ring)* if every simple (resp., simple singular) left R-module is f-injective. These rings generalize both left V-rings and regular rings simultaneously, and were characterized by Ming [7].

The purpose of this note is to study the connections between left f-V-rings and fully left idempotent rings (or regular rings), and we give some sufficient conditions for a left f-GV-ring to be a left f-V-ring. All rings in this note are associative rings with identity and all modules are unitary left modules.

We first need the following Lemma due to Ming [5]:

LEMMA 1. *Let R be a left f-V-ring. Then R is fully left idempotent (i.e. $I^2 = I$ for each left ideal I of R).*

Now we prove the following.

LEMMA 2. *Every factor ring of a left f-V-ring is also a left f-V-ring.*

Proof. Let $\bar{R} = R/I$ for some ideal I of R and let S be a simple left $\bar{R}$-module. Then S is simple as a left R-module satisfying $\mathrm{Ann}_R(S) \supset I$. Hence S is f-injective. Now let $\bar{f} : (\bar{x}_1, ..., \bar{x}_n) \to S$ be a nonzero homomorphism, where $\bar{x}_i \in \bar{R}$ are all distinct for $i = 1, ..., n$. For each $i = 1, ..., n$, choose an element x_i of R whose canonical image in $\bar{R}$ is $\bar{x}_i$. Then $f : (x_1, ..., x_n) \to S$ defined by $f(x_i) = \bar{f}(\bar{x}_i)$ is an R-homomorphism. Therefore there exists an element $y \in S$ such that

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$f(x_i) = x_i y$. Hence $\bar{f}(\bar{x}_i) = \bar{x}_i y$ so that S is f-injective as a left $\bar{R}$-module.

We recall that a ring R is said to be *left semi-artinian* if each nonzero left R-module contains a simple submodule. From Lemma 1 and Theorem 17 [2], we can see that the following conditions are equivalent for a semi-artinian ring R : (i) R is fully idempotent ; (ii) R is regular ; (iii) R is a left f-V-ring. We note that semi-artinian left V-rings are regular, but semi-artinian regular rings need not be left V-rings (See Example 1 [2]). For arbitrary rings, we prove the following two theorems about connections between left f-V-rings and other rings.

THEOREM 3. *A ring R is a left f-V-ring if and only if R is fully left idempotent and R/P is a left f-V-ring for every left primitive ideal P of R.*

Proof. The sufficiecy follows from Lemma 1 and 2. To prove the necessity, let S be a simple left R-module, I a finitely generated left ideal of R and $f : I \to S$ a nonzero homomorphism. If we let $P = \mathrm{Ann}_R(S)$, then P is a left primitive two-sided ideal and hence $P \bigcap I = PI$ by Theorem 8 [2]. Hence f can be extended to a homomorphism $g : I+P \to S$ such that $g(a+p) = f(a)$ for every $a \in I$ and $p \in P$.
Since $(I+P)/\mathrm{Ker}\ g$ can be regarded as a left R/P-module, $(I+P)/\mathrm{Ker}\ g \simeq^{\alpha} S$ is f-R/P-injective by hypothesis. Now consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & (I+P)/P & \longrightarrow & R/P \\ & & \Big\downarrow{\scriptstyle \alpha\circ j} & & \\ & & S & & \end{array}$$

where $j : (I+P)/P \to (I+P)/\mathrm{Ker}\ g$ is the induced map from 1_{I+P}. Since $(I+P)/P \simeq I/(I \bigcap P)$ is also finitely generated, there exists a homomorphism $h : R/P \to S$ such that $h \circ \bar{i} = \alpha \circ j$, where $\bar{i} : I+P \to R/P$ is the induced map from the canonical injection $i : I+P \to R$. Let $p : R \to R/P$ be the canonical projection. Then $p \circ i = \bar{i} \circ p$ and hence $(hp)i = h\bar{i}p = \alpha jp = g$. Hence $h \circ p$ is an extension of f. Thus R is a left f-V-ring.

The next result generalizes Theorem 13 of [2].(If R is a left V-ring in which every prime factor ring of R is regular, then R is regular.)

THEOREM 4. *A ring R is von Neumann regular if and only if R is a left f-V-ring and every prime factor ring of R is regular.*

Proof. If a ring R is regular, then every left R-module is f-injective since every finitely generated left ideal I of R is a direct summand of R so that every nonzero homomorphism f from I can be extended to a homomorphism from R. Thus a regular ring is a left f-V-ring. For the proof of the necessity, let $\bar{R}$ be a factor ring of R. Then $\bar{R}$ is a left f-V-ring by Lemma 2 and hence $\bar{R}$ is fully left idempotent by Lemma 1. Thus every factor ring of R is semiprime. By Corollary 1.3 [3], R is regular.

Recall that a ring R is a left f-GV-ring if every simple singular left R-module is f-injective; that is, each simple left R-module is either projective or f-injective. It is easy to see that a left f-GV-ring has projective left socle and a left f-GV-ring with zero socle is a left f-V-ring. Now we give more sufficient conditions for a left f-GV-ring to be a left f-V-ring.

PROPOSITION 5. *A ring R is a left f-V-ring if and only if R is a left f-GV-ring and every finitely generated left ideal is an intersection of maximal left ideals.*

Proof. The sufficiency follows from Theorem 6 [6]. For the necessity, let S be a simple left R-module which is not singular. Then S is projective by Proposition 1.24 [4], and hence S is finitely presented. If we let I be a finitely generated left ideal of R with a nonzero homomorphism $f : I \rightarrow S$, then Ker f is finitely generated and it is an intersection of maximal left ideals of R. Hence there exists a maximal left ideal J of R such that Ker $f \subseteq J$ but $I \not\subseteq J$. Since Ker f is maximal in I, it follows that $I \bigcap J =$ Ker f. So $R/J = (I + J)/J \simeq I/(I \cap J) = I/\text{Ker } f \simeq S$, and therefore f can be extended to a homomorphism $g : R \rightarrow S$. Thus S is f-injective.

PROPOSITION 6. *If R is a left f-GV-ring in which every primitive idempotent is central, then R is a left f-V-ring.*

Proof. Let S be a simple projective left R-module, so that $S \simeq Re$

for a primitive idempotent e. If I is a finitely generated left ideal of R and $f : I \to S$ is a nonzero homomorphism, then $I = \text{Ker } f \oplus T$ for some left ideal T of R which is isomorphic to Re. Since e is central, Re is fully invariant and hence $Re = T$. Thus $R = T \oplus R(1-e)$, so Ker $f \subseteq R(1-e)$. Hence f can be extended to a homomorphism $g : R \to S$ such that $g = f$ on Re and $g = 0$ on $R(1-e)$.

The following is an easy consequence of Proposition 6, which characterizes commutative f-GV-rings.

COROLLARY 7. *For any commutative ring R, the following are equivalent :*

(1) R is a f-GV-ring,
(2) R is a f-V-ring,
(3) R is a fully idempotent ring,
(4) R is a von Neumann regular ring.

References

1. C. Faith, *Lectures on injective modules and quotient rings*, Lecture Notes in Math., No. 49, Springer-Verlag, Berlin, 1967.
2. J. W. Fisher, *Von Neumann regular rings versus V-rings*, Ring Theory : Proc. Univ. Oklahoma Conference, Lecture Notes in Pure and Appl. Math., No. 7, Dekker, New York (1974), 101–119.
3. J. W. Fisher and R. L. Snider, *On the von Neumann regularity of rings with regular prime factor rings*, Pac. J. Math. 54 (1974), 135–144.
4. K. R. Goodearl, *Ring Theory : Nonsingular rings and modules*, Pure and Appl. Math. Ser. 33, Dekker, New York, 1976.
5. R. Yue Chi Ming, *On simple p-injective modules*, Math. Japonica 19 (1974), 173–176.
6. R. Yue Chi Ming, *On generalizations of V-rings and regular rings*, Math. J. Okayama Univ. 20 (1978), 123–129.
7. R. Yue Chi Ming, *On V-rings and prime rings*, J. Algebra 62 (1980), 13–20.
8. H. Tominaga, *On s-unital rings*, Math. J. Okayama Univ. 18 (1976), 117–134.
9. H. Tominaga, *On s-unital rings II*, Math. J. Okayama Univ. 19 (1977), 171–182.

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