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RIEMANN-STIELTJES INTEGRAL OF FUNCTIONS OF κ-BOUNDED VARIATION

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Let $\{I_i\}$ be a collection of nonoverlapping subintervals of $[a,b]$ which covers $[a,b]$. A function is of *bounded variation* on $[a,b]$ if $V(f) = \sup\sum|f(I_i)| < \infty$ where $f(I_i) = |f(x_{i-1}) - f(x_i)|$, $I_i = [x_{i-1}, x_i]$. Cyphert [1] generalized this idea by considering other functions κ on $[a,b]$. The introduction of the function κ can be viewed as a rescaling of lengths of subintervals of $[a,b]$ such that the length of $[a,b]$ is 1 if $\kappa(1) = 1$. In the sequel, we require that κ-function have the following properties on $[0,1]$:

(1) κ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$,

(2) κ is concave and strictly increasing, and

(3) $\displaystyle\lim_{x\to 0+}\frac{\kappa(x)}{x} = \infty$.

A function f is said to be of *κ-bounded variation* on $[a,b]$ if there exists a positiove constant C such that for every collection $\{I_i\}$ of non-everlapping subintervals of $[a,b]$, $\sum|f(I_i) \le C\sum\kappa(\frac{|I_n|}{b-a})$, where $|I_n|$ is the length of the interval I_n. The total κ-variation of f over $[a,b]$ is defined by $\kappa V(f) = \sup\frac{\sum|f(I_i)|}{\sum\kappa(\frac{|I_i|}{b-a})}$, where the supremum is taken over all $\{I_i\}$ of nonoverlapping subintervals of $[a,b]$ which cover $[a,b]$. Since κ is subadditive, every function f of bounded variation is of κ-bounded variation and $\kappa V(f) \le V(f)$. Also f has at most a countable number of points of simple discontinuity [1]. Although functions of κ-bounded variation are not necessarily of bounded variation, they do remain bounded.

Let κBV be the set of functions of κ-bounded variation on the closed interval $[a,b]$ and define for each f in κBV

$$\|f\|_\kappa = \kappa V(f) + |f(a)|.$$

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Then $\| \bullet \|_\kappa$ is a norm on κBV and κBV is a Banach space under this norm. Schramm [3] shows the existence of Riemann–Stieltjes integral of functions of Φ–bounded variation. We will show the existence of the Riemann–Stieltjes integral of functions of κ–bounded variation.

For each n, put

$$\kappa V(n,f) = \sup \frac{\sum_i^n |f(I_i)|}{\sum_{i=1}^n \kappa(\frac{|I_i|}{b-a})}$$

over all collections $\{I_i\}$ consisting of nonoverlapping n intervals of $[a,b]$ such that $[a,b] = \cup_{i=1}^n I_i$. Then

$$\kappa V(f) = \sup_n \kappa V(n,f).$$

Let $\{I_i\}_{i=1}^\infty$ be a sequence of nonoverlapping intervals of $[a,b]$ which covers $[a,b]$. We call $\{I_i'\}_{i=1}^\infty$ the decreasing rearrangerent of $\{I_i\}_{i=1}^\infty$ with respect to f if $|f(I_{i+1}')| \le |f(I_i')|$ for all i.

For the simplicity of notation, we will express $\kappa(|I_i|) = \kappa(\frac{|I_i|}{b-a})$.

Define $I_\kappa(f) = \sup_n \frac{\sum_{i=1}^n |f(I_i')|}{\sum_{i=1}^n \kappa(|I_i'|)}$. It is clear that if $\{I_i''\}$ is a rearrangenent of $\{I_i\}$, then

$$\sum_1^n |f(I_i'')| \le I_\kappa(f) \sum_{i=1}^n \kappa(|I_i'|), \text{ for all } n.$$

In the sequel, $\{I_i'\}$ and $\{J_i'\}$ denote the decreasing rearrangement of $\{I_i\}$ and $\{J_i\}$ with respect to f and g, respectively and k_i are κ-functions.

LEMMA 1. *Let $I = \{I_i\}$ and $J = \{J_i\}$ be sequences as above and $f \in \kappa_1 BV$, $g \in \kappa_2 BV$. Then for each n, there is $k \le n$ such that*

$$|f(I_k)g(J_k)| \le \frac{\sum_{i=1}^n \kappa_1(|I_i'|) \sum \kappa_2(|J_i'|)}{n^2} I_{\kappa_1}(f) J_{\kappa_2}(g).$$

Proof. There is k such that $|f(I_k)g(J_k)|$ is not larger then the geometric mean of the numbers $|f(I_1)g(J_1)|, \dots, |f(I_n)g(J_n)|$. Thus we have

$$\begin{aligned} |f(I_k)g(J_k)| &\le (\frac{1}{n}\sum_{i=1}^n |f(I_i)|)(\frac{1}{n}\sum_{i=1}^n |g(J_i)|) \\ &\le \frac{\sum \kappa_1(|I_i'|) \sum \kappa_2(|J_i'|)}{n^2} I_{\kappa_1}(f) J_{\kappa_2}(g). \end{aligned}$$

LEMMA 2. *Let I and J be sequences as above and $f \in \kappa_1 BV$, $g \in \kappa_2 BV$. Then*

$$\sum_{k=1}^{n} |f(I_k)g(J_k)| \leq \sum_{k=1}^{n} \left(\frac{\sum_{i=1}^{k} \kappa_1(|I_i'|) \sum \kappa_2(|J_i'|)}{k^2}\right) I_{\kappa_1}(f) J_{\kappa_2}(g).$$

Proof. Assume that $|f(I_k)g(J_k)|$ are arranged in decreasing order, then

$$|f(I_k)g(J_k)| \leq \frac{\sum^k \kappa_1(|I_i'|) \sum^k \kappa_2(|J_i'|)}{k^2} I_{\kappa_1}(f) J_{\kappa_2}(g).$$

Thus, the conclusion follows.

Note that if $\displaystyle\sum_{k=1}^{\infty} \frac{\sum_{i=1}^{k} \kappa_1(|I_i'|) \sum_{i=1}^{k} \kappa_2(|J_i'|)}{k^2} = M < \infty$, then we will have

$$\sum_{k=1}^{\infty} |f(I_k')g(J_k')| \leq M I_{\kappa_1}(f) J_{\kappa_2}(g).$$

By a partition T on the finite sequence $I = \{I_1, \ldots, I_n\}$, we shall mean a sequence $TI = \{J_1, \ldots, J_r\}$, $r \leq n$ where each J_i are the union of some consecuvtive intervals I_i'.

LEMMA 3. *If $f \in \kappa_1 BV$, $g \in \kappa_2 BV$, then*

$$\sum_{k=1}^{n} \sum_{i=1}^{k} |f(I_i)||g(I_k)|$$
$$\leq (1 + \sum_{k=1}^{n-1} \left(\frac{\sum_{i=1}^{k} \kappa_1(|I_i'|) \sum_{i=1}^{k} \kappa_2(|J_i'|)}{k^2}\right)) \sup TI_{\kappa_1}(f) QJ_{\kappa_2}(g)$$

where the supremums are taken over all partitions T and Q.

Proof. For a fixed $k \leq n$, let T_k be the operation defined by

$$\begin{aligned} T_k I &= \{I_1, \ldots, I_{k-1},\ I_k \cup I_{k+1}, I_{k+2}, \ldots, I_n\} \\ &= \{I_1', I_2', \ldots, I_{n-1}'\} \\ &= I'. \end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{i=1}^{n}(|f(I_1)|+\cdots+|f(I_i)|)|g(J_i)| \\
&\leq \sum_{i=1}^{n-1}(|f(I'_1)|+\cdots+|f(I'_i)|)|g(J'_i)|-|f(I_{k+1})||g(J_k)| \\
&\leq \sum_{k=1}^{n-1}\sum_{i=1}^{k}|f(I'_i)g(J'_i)|+|f(I_{k+1})g(J_k)|
\end{aligned}$$

By Lemma 1, we may choose k in such a way that

$$\begin{aligned}
|f(I_{k+1})g(J_k)| &\leq \frac{1}{(n-1)^2}\sum_{2}^{n}|f(I_i)|\sum_{1}^{n-1}|g(J_i)| \\
&\leq I_{\kappa_1}(f)K_{\kappa_2}(g)\frac{\sum\kappa_1(|I'_i|)\sum\kappa_2(|J'_i|)}{(n-1)^2}
\end{aligned}$$

Following the same procedure we see that

$$\begin{aligned}
&\sum_{k=1}^{n-1}\sum_{i=1}^{k}|f(I'_i)g(J'_i)| \\
&\quad\leq \sum_{k=1}^{n-2}\sum_{i=1}^{k}|f(I''_i)g(J''_i)+I'_{\kappa_1}(f)J'_{\kappa_2}(g)\frac{\sum_{i=1}^{n-2}\kappa_1(I''_i)\sum_{i=1}^{n-2}\kappa_2(J''_i)}{(n-2)^2}
\end{aligned}$$

Where $I''=\{I''_1,\ldots,I''_{n-2}\}$, $J''=\{J''_1,\ldots,J''_{n-2}\}$ are sequence of length $n-2$ obtained from I and J by $T_j\circ T_k$. Continuing this procers, we obtain the desired inequality.

LEMMA 4. *Let* $\lim\frac{\sum_{i=1}^{n}\kappa_1(|I_i|)}{\sum_{i=1}^{n}\kappa(|I_i|)}=\infty$. *Given* $\varepsilon>0$ *and* $A>0$, *there is an* $\eta>0$ *such that* $I_{\kappa_1}(f)\leq\varepsilon$ *for all* $\{I_i\}$ *such that* $I_\kappa(f)\leq A$ *and* $|f(I_i)|<\eta$ *for all* i.

Proof. Let m be so large that

$$\frac{\sum^{n}\kappa_1(|I_i|)}{\sum\kappa(|I_i|)}<\frac{\varepsilon}{2A}\text{ if }n\geq m.$$

Choose a positive

$$\eta < \frac{\varepsilon}{2m}.$$

If $I_\kappa(f) \le A$, $|f(I_i)| < \eta$ for all i, and $\{I_i'\}$ is the decreasing rearrangement of $\{I_i\}$, then we have for $n \ge m$,

$$\sum_{i=1}^{n} |f(I_i')| = \sum_{i=1}^{m-1} |f(I_i')| + \sum_{i=m}^{n} |f(I_i')|$$
$$\le \frac{m\varepsilon}{2m} + \frac{\varepsilon A \sum_{i=1}^{n} \kappa_1(|I_i'|)}{2A}$$

If $n < m$, then

$$\sum_{i=1}^{n} |f(I_i')| \le \frac{n\varepsilon}{2m} \le \frac{\sum_{i=1}^{n} \kappa_1(|I_i'|)}{2}$$

Therefore $I_{\kappa_1}(f) \le \varepsilon$.

We now prove the main result of this paper.

THEOREM 5. *If $f \in \kappa_1 BV$, $g \in \kappa_2 BV$, $\sum_{n=1}^{\infty} \left(\frac{\sum_{i=1}^{n} \kappa_1(|I_i|) \sum_{i=1}^{n} \kappa_2(|J_i|)}{n^2} \right)$ $< \infty$ and f, g have no common discontinuity, then the Riemann–stieltjes integral $\int_a^b f dg$ exists.*

Proof. f has only simple discontinuities at $t_1, t_2, \ldots$. May assume f is right continuous at a and left continuous at b. Let

$$d_i = \max\{|f(t_i+0) - f(t_i)|,\ |f(t_i-0) - f(t_i)|,\ |f(t_i+0) - f(t_i-0)|\}.$$

Given $\varepsilon > 0$, choose $\eta < \varepsilon$ as in Lemma 4 and $A = 2\kappa_1 V(f)$. Since

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \kappa_1(|I_i|)}{n} = 0, \lim \frac{\sum_{i=1}^{n} |f(I_i)|}{n} = 0.$$

Thus $d_i \to 0$ as $i \to \infty$ and hence there is N such that $d_i < \frac{\eta}{4}$ for $i > N$. There is $\kappa_\alpha(x) = x^\alpha$ $(0 < \alpha < 1)$ such that $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \kappa_\alpha(|J_\kappa|)}{\sum_{i=1}^{n} \kappa_2(|J_\kappa|)} = \infty.$

Then we apply again Lemma 4 again, choosing η_1 such that $H_{\kappa_\alpha}(g) < \varepsilon/N$ for any sequence $H = \{H_i\}_{i=1}^{\infty}$ with $H_{\kappa_2}(g) \le \kappa_2 V(g)$ and $|f(H_i)| < \eta_1$ for all i. Let $\delta > 0$ be less than the minimum distance between two of the set $\{a, t_1, t_2, \dots, t_N, b\}$ such that

i) $|f(x) - f(y)| < \frac{\eta}{2}$ whenever $|x-y| < \delta$ and $[x,y]$ does not contain any of the t_i, $i = 1, 2, \dots, N$.

ii) $|g(x) - g(y)| < \min\{\varepsilon/N, \eta_1\}$ whenever $[x,y] \subseteq [t_i - \delta, t_i + \delta]$ for some $i = 1, 2, \dots, N$.

Let $P_1 = \{[x'_{k-1}, x'_k]\}_{k=1}^{n_1}$ and $P_2 = \{[x^2_{k-1}, x^2_k]\}_{k=1}^{n_2}$ be two partitions of $[a,b]$ with sets of intermediate points $Q_1 = \{\xi^1_k\}$ and $Q_2 = \{\xi^2_k\}$ respectively, with mesh less than $\frac{\delta}{2}$. May assume all end points are different from t_i, $i = 1, 2, \dots, N$. Define step functions f_1, and f_2 by

$$f_i(x) = \begin{cases} \{f(\xi^i_1), & x = a \\ f(\xi^i_k), & \xi^i_k \in (x^i_{k-1}, x^i_k] \end{cases}$$

for $i = 1, 2$. Let $P = \{[x_{k-1}, x_k]\}_{k=1}^{n}$ be the common refinement of P_1 and P_2. Then we have the Riemann–Stieltjes sum of f with respect to g corresponding to P_i and Q_i, $i = 1, 2$,

$$\begin{aligned} S(f, g; P_i, Q_i) &= \sum_{i=1}^{n_i} f(\xi^i_k)(g(x^i_k) - g(x^i_{k-1})) \\ &= \sum_{i=1}^{n_i} f_i(x^i_k)(g(x^i_k) - g(x^i_{k-1})) \\ &= \sum_{k=1}^{n} f_i(x_k)(g(x_k) - g(x_{k-1})), \end{aligned}$$

so that

$$S(f, g; P_1, Q_1) - S(f, g; P_2, Q_2) = \sum_{k=1}^{n} (f_1(x_k) - f_2(x_k))(g(x_k) - g(x_{k-1})).$$

Let σ be the subcollections of P which contain some t_i, $i = 1, 2, \dots, N$

and let σ' be the subcollections of P which do not contain any t_i. Then

$$\begin{aligned}
&|S(f,g;P_1,Q_1)-S(f,g;P_2,Q_2)|\\
\leq&|\sum_{I_k\in\sigma}(f_1(x_k)-f_2(x_k))(g(x_k)-g(x_{k-1}))|\\
&\quad+|\sum_{I_k\in\sigma}(f_1(x_k)-f_2(x_k))(g(x_k)-g(x_{k-1}))|\\
&\text{where}\quad I_k=[x_{k-1},x_k]\\
=&I+II.
\end{aligned}$$

From ii) we have that

$$I\leq 2\sup|f(x)|\cdot\frac{N\varepsilon}{N}=2\sup|f(x)|\varepsilon.$$

Denote by $[v_1,u_1],[v_2,u_2],\ldots,[v_s,u_s]$ the intervals in σ', ordered from left to right. To estimate II, we observe that $v_1=a$, $u_s=b$, and $v_i\neq u_{i-1}$ if and only if v_i and u_{i-1} are respectively the right and left end point of an interval in σ. Let $f_3=f_1-f_2$, and $u_0=a$. Then

$$\begin{aligned}
&|\sum_{I_k\in\sigma'}(f_1(x_k)-f_2(x_k)(g(x_k)-g(x_{k-1}))|\\
=&|\sum_1^s(f_1(u_i)-f_2(u_i))(g(u_i)-g(v_i))|\\
\leq&|\sum_1^s f_3(u_i)(g(u_i)-g(u_{i-1}))|+|\sum_1^s f_3(u_i)(g(v_i)-g(u_{i-1}))|\\
\leq&|\sum_1^s f_3(u_i)(g(u_i)-g(u_{i-1}))|+2\sup|f(x)|\sum_{I_k\in\sigma}|g(I_k)|\\
\leq&|\sum_1^s f_3(u_i)(g(u_i)-g(u_{i-1}))|+2\sup|f(x)|\frac{N\varepsilon}{N}
\end{aligned}$$

Let us estimate $III = |\sum_1^s f_3(u_i)(g(u_i)-g(u_{i-1}))|$. Put $H_j = [u_{j-1}, u_j]$.

$$III = |\sum_{i=1}^{s}(f_3(a) + \sum_{j=1}^{i} f_3(H_j))|g(u_i) - g(u_{i-1})|$$

$$\leq |f_3(a)||g(b) - g(a)| + \sum_{i=1}^{s}\sum_{j=1}^{i} |f_3(H_j)|\,|g(H_i)|.$$

By Lemma 3,

$$\sum_{i=1}^{s}(\sum_{j=1}^{i} |f_3(H_j)|)|g(H_i)|$$

$$\leq (1 + \sum_{i=1}^{n-1}\left(\frac{\sum_{j=1}^{i}\kappa_1(|H_i'|)\sum_{j=1}^{i}\kappa_2(|H_j'|)}{i^2}\right))\sup_T TH_{\kappa_1}(f_3)\kappa_2 V(g).$$

But $f_3(a) = f(\xi_1) - f(\xi_1')$ and therefore, by i), $|f_2(a)| < \frac{\eta}{2} < \varepsilon$. Each interval of TH is of the form $[u_{j_{l-1}}, u_{j_l}]$ for some subsequence $j_0 < \cdots < j_t$ of $\{1, \ldots, s\}$. Then $TH_{\kappa_1}(f_3) \leq \kappa_1 V(f_3) \leq \kappa_1 V(f_1) + \kappa_1 V(f_2) \leq 2\kappa_1 V(f)$. Also $|f_3(H_{j_l})| = |f_3(u_{jl}) - f_3(u_{jl-1})| \leq |f_3(u_{jl})| + |f_3(u_{jl-1})| < \eta/2 + \eta/2$ by i), since both $f_3(u_{jl})$ and $f_3(u_{jl-1})$ are differences of values of f at ξ_i^1 and ξ_i^2 where $|\xi_i^1 - \xi_i^2| < \delta$ and $[\xi_i^1, \xi_i^2] \not\ni t_k$, $k = 1, \ldots, N$. Hence by the Lemma 4, $\sup_T TH_{\kappa_1}(f_3) < \varepsilon$ and thus

$$II \leq \varepsilon\left[|g(b) - g(a)| + \kappa_2 V(g)(1 + \sum_{i=1}^{\infty}\left(\frac{\sum^i \kappa_1(|H_j'|)\sum^i \kappa_2(|H_j'|)}{i^2}\right))\right]$$

We have proved that for any $\varepsilon > 0$, there are partitions P_i, Q_i $(i = 1, 2)$ such that $|S(f, g; P_1, Q_1) - S(f, g; P_2, Q_2)| < \varepsilon$. Thus $\int_a^b f dg$ exists.

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