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A NOTE ON THE CALDERON-ZYGMUND SINGULAR INTEGRAL OPERATORS

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1. Introduction

Bruna and Korenblum considered the problem of finding the necessary and sufficient condition on a nonnegative function ψ for the existence of $f \in L^1$ such that $\psi \leq |Tf|$ a.e., where T is the Calderon-Zygmund singular integral operator, and found a strong necessary condition which is $\|M_{1\infty}^m \psi\|_{1\infty} \leq c_1 c_2^m$. In fact, they proved in [1] that $M_{1\infty}^m T^* f \in L^{1\infty}$ for all $f \in L^1$ using the fact $M_{1\infty} Mf(x) \leq cMf(x)$, where $Mf(x)$ is the Hardy-Littlewood maximal function of f. (For the definitions, see section 2.)

In this paper, by a generalization of $M_{1\infty} Mf(x) \leq cMf(x)$, we generalize Bruna and Korenblums theorem in [1] for any L^p case, $p \geq 1$.

2. Main Result

We consider the following singular integral operators.

Let K be a kernel in R^n of class C^1 outside the origin satisfying

$$|K(x)| \leq c|x|^{-n}$$
$$|\nabla K(x)| \leq c|x|^{-n-1}$$

For $\epsilon > 0$ and $f \in L^p(R^n)$, $1 \leq p < \infty$, set

$$T_\epsilon f(x) = \int_{|y|>\epsilon} f(x-y)K(y)dy$$

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and

$$Tf(x) = \lim_{\epsilon \to 0} T_\epsilon f(x)$$
$$T^* f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|$$

We assume that K satisfies the usual properties ensuring that the mapping $f \mapsto T^* f$ is of weak type (1,1) and of type (p,p) for $1 < p < \infty$ and hence for $f \in L^p$, $p \geq 1$, $Tf(x)$ exists for a.e. x. (For the conditions, refer to [7] chaper II.)

$L^{p,q}$ is the Lorentz space with the "norm"

$$\|f\|_{pq} = \begin{cases} [\int_0^\infty \lambda_f(y)^{\frac{q}{p}} d(y^q)]^{\frac{1}{q}}, & 1 \leq q < \infty \\ \sup_{y>0} y[\lambda_f(y)]^{\frac{1}{p}}, & q = \infty \end{cases}$$

where $\lambda_f(y) = |\{x : |f(x)| > y\}|$ is the distribution function of f. For the properties of $\| \cdot \|_{pq}$ we refer to [4].

Let M_{pq} be the operator defined by

$$M_{pq} f(x) = \sup_B \frac{\|f \cdot \chi_B\|_{pq}}{\|\chi_B\|_{pq}}$$

where the supremum is taken over all balls B centered at x. The notation $M_{pq}^m f(x)$ stands for the function obtained by applying m times the operator M_{pq}, whenever it makes sense.

We have the following iteration theorem for maximal functions.

LEMMA 1. *For* $1 \leq p \leq r < \infty$, $1 \leq s \leq r$,

$$M_{p\infty} M_{rs} f(x) \leq c M_{rs} f(x), \quad x \in R^n$$

where c is a constant depending on n and r.

Proof. In [5] we proved the same result with cubes containing x in place of balls centered at x. The proof is essentially the same except that we use the Besicovitch covering lemma [3] instead of the one in [5]. We omit the proof.

REMARK. *C. J. Neugebauer proved the above for the dyadic case. [6]*

Write $M_p f(x)$ for $M_{pp} f(x)$. Then lemma 1 implies, for $1 \le p < \infty$,

$$M_{p\infty} M_p f(x) \le c M_p f(x), \quad x \in R^n.$$

Using Jensen's inequality we can easily show the following.

LEMMA 2. $M_1 f(x) \le M_p f(x)$ *for* $p \ge 1$.

If we note that $M_p f = [M(|f|^p)]^{\frac{1}{p}}$ and the Hardy-Littlewood maximal function is of weak type (1,1), we have the following.

LEMMA 3. *If* $f \in L^p$, $\|M_p f\|_{p\infty} \le c\|f\|_p$ *and hence* $M_p f \in L^{p\infty}$.

Using the above lemmas we can prove our theorem by modifying the proof in [1].

THEOREM. *For any* $1 \le p < \infty$, $M^m_{p\infty}(T^* f) \in L^{p\infty}$ *for all* $f \in L^p$ *and all* $m \in N$. *Moreover,*

$$\|M^m_{p\infty}(T^* f)\|_{p\infty} < c_1 c_2^m \|f\|_p.$$

Proof. Fix $p \ge 1$. We will show

$$(*) \qquad M_{p\infty}(T^* f) \le c\{T^* f + M_p f\}.$$

Then since T^* is of weak-type (p, p), by lemma 3, $M^m_{p\infty}(T^* f) \in L^{p\infty}$. Also the norm inequality can be shown by the straight-forward computation. Fix x and let B be a ball centered at x. Let $2B$ denote the ball with the same center as B and the radius twice that of B.

Consider $f_1 = f \cdot \chi_{2B}$ and $f_2 = f - f_1$. Then $T^* f \le T^* f_1 + T^* f_2$ and so

$$\|T^* f \cdot \chi_B\|_{p\infty} \le 2(\|T^* f_1 \cdot \chi_B\|_{p\infty} + \|T^* f_2 \cdot \chi_B\|_{p\infty}).$$

Since T^* satisfies the weak-type (p,p) inequality, we have

$$
\begin{aligned}
(1) \qquad \|T^* f_1 \cdot \chi_B\|_{p\infty} &\le \|T^* f_1\|_{p\infty} \\
&\le c\|f_1\|_p \\
&= c[\int_{2B} |f(y)|^p dy]^{\frac{1}{p}} \\
&= c|2B|^{\frac{1}{p}} \frac{[\int_{2B} |f(y)|^p dy]^{\frac{1}{p}}}{|2B|^{\frac{1}{p}}} \\
&= c|2B|^{\frac{1}{p}} M_p f(x)
\end{aligned}
$$

Now as shown in [1] for $z \in B$,

$$|T^* f_2(z)| \le c'(T^* f(x) + M f(x)).$$

From lemma 2, therefore,

$$|T^* f_2(z)| \le c'(T^* f(x) + M_p f(x)) \text{ for } z \in B$$

and hence

$$
\begin{aligned}
(2) \qquad \|T^* f_2 \cdot \chi_B\|_{p\infty} &\le \sup_{y>0} y \left([\frac{c'(T^* f(x) + M_p f(x))}{y}]^p |B| \right)^{\frac{1}{p}} \\
&= |B|^{\frac{1}{p}} c'(T^* f(x) + M_p f(x)).
\end{aligned}
$$

Thus, from (1) and (2),

$$
\begin{aligned}
\|T^* f \cdot \chi_B\|_{p\infty} &\le 2[c|2B|^{\frac{1}{p}} M_p f(x) \\
&\quad + c'|B|(T^* f(x) + M_p f(x))] \\
&\le c^* |B|^{\frac{1}{p}} (T^* f(x) + M_p f(x))
\end{aligned}
$$

where c^* is a constant depending on n and p. Therefore,

$$M_{p\infty} T^* f(x) \le c^*(T^* f(x) + M_p f(x))$$

and hence we have the inequality (*).

Since T^* is of weak type (p,p), by lemma 3, (*) implies that $M_{p\infty}T^*f \in L^{p\infty}$ if $f \in L^p$.

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