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STEIN NEIGHBORHOOD BASES FOR PRODUCT SETS IN $\mathbf{C}^n$ *

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1. Introduction

The complex analytic properties of bounded pseudoconvex domains with smooth boundaries in $\mathbf{C}^n$ can differ very much from those of strictly pseudoconvex domains. The existence of pseudoconvex neighborhoods was shown by K. Diederich-J. E. Fornaess [5] under the assumption that $E = M_1 \cup M_2 \cup \cdots \cup M_k$ is the union of submanifolds with a nondegeneracy condition. If E contains a complex submanifold, this nondegeneracy condition is not fulfilled, and in fact a Stein neighborhood system need not exist in general, as was shown by K. Diederich-J. E. Fornaess [4]. And K. Diederich-J. E. Fornaess [4] proved that if $\Omega \subset\subset \mathbf{C}^2$ is a pseudoconvex domain with C^3-boundary and such that the set M of degeneracy of the Levi form is exactly the disc $M = \{(z,w); |z| \leq 1,\ w = 0\}$ then Ω has a Stein neighborhood basis. In case that the boundary of the domain $\Omega \subset\subset \mathbf{C}^n$ is smooth real analytic, Ω has a Stein neighborhood basis by K. Diederich-J. E. Fornaess [6]. Also, E. Bedford-J. E. Fornaess [1] obtained assorted fundamental results and investigated pseudoconvex neighborhood systems. Y. T. Siu [21] showed that every Stein subvariety admits a Stein neighborhood. Recently H. Kazama [11] proved that $\mathbf{C}^m \times \mathbf{R}^n$ has no Stein neighborhood bases in $\mathbf{C}^m \times \mathbf{C}^n$ for all $m, n \geq 1$.

Let $\Delta = \{z \in \mathbf{C}; |z| < 1\}$ be the unit open disc in the complex plane $\mathbf{C}$, $\overline{\Delta}$ its closure and T its boundary. In the preceding paper [19], the author has shown that there are no Stein neighborhood bases of the product sets $\Delta \times \overline{\Delta}$ and $\Delta \times T$ in $\mathbf{C}^2$, and more generally that the product set $R_1 \times \overline{R_2}$ of Reinhardt Stein domains $R_1 \subset \mathbf{C}^m$ and $R_2 \subset\subset \mathbf{C}^n$ containing

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the origins has no Stein neighborhood bases in $\mathbf{C}^m \times \mathbf{C}^n$. Also in [20], the ahthor proved that the product set $P \times (a, b)$ of an open polydisc P and an open interval (a, b) has no Stein neighborhood bases in $\mathbf{C}^m \times \mathbf{C}$. H. Kazama [11] and L. C. Piccinini [17, 18] investigated the Cauchy-Riemann equations depending real analytically on a parameter. The author [19, 20] obtained similar results for the product sets $\Delta \times T$, $\Delta \times \mathbf{R}$ and $P \times (a, b)$.

In this paper we investigate properties of a Stein domain which is an open neighborhood of the product set $\Delta \times \mathbf{R}$ in $\mathbf{C}^2$ and investigate Stein neighborhood bases of the product set $\Omega \times (a, b)$ of a hyperbolic complex manifold Ω and an open interval (a, b).

2. Pseudoconvex domains and the Levi problem

E. E. Levi [12] showed that the boundary of a domain of holomorphy is not arbitrary. The boundary satisfies a condition of convexity called pseudoconvex. The pseudoconvexity of a domain is a local property of the boundary. A domain $\Omega \subset \mathbf{C}^n$ is said to have a C^j boundary $(j \geq 1)$ if there is a C^j function $\Phi : U \to \mathbf{R}$ on a neighborhood U of Ω such that $\Omega = \{z; \Phi(z) < 0\}$ and $\operatorname{grad} \Phi(z) \neq 0$ on the boundary $b\Omega$ of Ω. A domain of holomorphy is a domain on which there exists a holomorphic function which cannot be extended to a large domain.

DEFINITION 2.1. *A domain Ω in $\mathbf{C}^n$ is said to be C-pseudoconvex if, for any $z \in b\Omega$, there is a neighborhood U of z in $\mathbf{C}^n$ such that $U \cap \Omega$ is a domain of holomorphy.*

DEFINITION 2.2. *A real valued function $\Phi(z)$ of class C^2 is said to satisfy the Levi-Krzoska's condition at a point z^0 if for any pair of complex numbers $w_1, w_2, \cdots, w_n$ of which at least one is not zero, satisfying*

$$\sum_{j=1}^{n} \left(\frac{\partial \Phi}{\partial z_j}\right)_{(z^0)} w_j = 0,$$

we have

$$\sum_{j,k=1}^{n} \left(\frac{\partial^2 \Phi}{\partial z_j \partial \overline{z}_k}\right)_{(z^0)} w_j \overline{w_k} > 0.$$

The above Hermitian form $\sum_{j,k=1}^{n}(\frac{\partial^2\Phi}{\partial z_j\partial\overline{z_k}})_{(z^0)}w_j\overline{w_k}$ *is called the Levi form of* Φ *at* z^0.

DEFINITION 2.3. *A domain* Ω *in* $\mathbf{C}^n$ *with* C^2 *boundary is said to be* $(L-)$*pseudoconvex if it has a defining function* Φ *such that the Levi form of* Φ *at* z^0 *is positive semi-definite for all* $z^0 \in b\Omega$ *and* $w \in \mathbf{C}^n$ *satisfying* $\sum_{j=1}^{n}(\frac{\partial\Phi}{\partial z_j})_{(z^0)}w_j = 0$.

There are many definitions of pseudoconvexity. For a domain in $\mathbf{C}^n$, the definitions of pseudoconvexity are all equivalent (see [7, 10]). The original Levi's problem is to prove the converse that every domain with smooth pseudoconvex boundary is a domain of holomorphy. For special domains, the Levi's problem was solved by H. Behnke [2]. For general domains, the problem was first solved by K. Oka [15, 16]. In the case of general dimension n, the problem was solved at the same time independently by H. J. Bremermann [3] and F. Norguet [14] but for schlicht domains.

LEMMA 2.4([20]). *Let* $\Omega \subset\subset \mathbf{C}^2$ *be a domain with* C^2 *boundary and suppose that* $\Phi : \mathbf{C}^2 \to \mathbf{R}$ *is of* C^2 *on the open neighborhood* U *of the boundary* $b\Omega$ *in* $\mathbf{C}^2$. *Then* Ω *is pseudoconvex if and only if*

$$L(\Phi)_{(z^0,w^0)} := -\begin{vmatrix} 0 & \frac{\partial\Phi}{\partial z} & \frac{\partial\Phi}{\partial w} \\ \frac{\partial\Phi}{\partial\overline{z}} & \frac{\partial^2\Phi}{\partial z\partial\overline{z}} & \frac{\partial^2\Phi}{\partial w\partial\overline{z}} \\ \frac{\partial\Phi}{\partial\overline{w}} & \frac{\partial^2\Phi}{\partial z\partial\overline{w}} & \frac{\partial^2\Phi}{\partial w\partial\overline{w}} \end{vmatrix}_{(z^0,w^0)} \geq 0$$

for all $(z^0, w^0) \in b\Omega$.

Let $L(\Phi)$ be the differential form of Lemma 2.4. By x, y, u, v, we denote the real coordinates such that $z = x + \sqrt{-1}y$ and $w = u + \sqrt{-1}v$. We set $\Phi(z, w) = \Phi(r\exp(\sqrt{-1}\theta),\ u + \sqrt{-1}v)$ for a nonzero complex number $z = x + \sqrt{-1}y$ and the Laplacians $\Delta_z = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}$ and $\Delta_w = \frac{\partial^2}{\partial^2 u} + \frac{\partial^2}{\partial^2 v}$ in the space $\mathbf{R}^2$.

LEMMA 2.5. *Let $U(u)$ be a positive C^2 function with $U(u) \leq \exp(-u^2)$ in $-\infty < u < \infty$. Then there exists a sequence $\{u_n\}$ of real numbers satisfying $U'(u_n) \to 0$ as $n \to \infty$.*

Proof. Assume that, for any sequence $\{u_n\}$, $U'(u_n)$ does not converge to 0 even if $n \to \infty$. We suppose that there is a positive number ε satisfying

$$\varliminf_{n \to \infty} |U'(u_n)| = \varepsilon.$$

By the mean value theorem, for any positive integer m, there exists a number $u_m \in (m, m+1)$ such that

$$\begin{aligned} |U'(u_m)| &= |U(m) - U(m+1)| \\ &\leq \exp(-m^2) + \exp(-(m+1)^2) \\ &\leq 2\ \exp(-m^2). \end{aligned}$$

For $m \to \infty$, we have $\exp(-m^2) \to 0$. Therefore, we have $\varepsilon \leq 0$. This is a contradiction.

LEMMA 2.6. *Let $U(u)$ be a positive C^2 function with $U(u) \leq \exp(-u^2)$ for $-\infty < u < \infty$. Then the following statement does not hold :*

There exists a real number a such that $U''(u) \leq 0$ for all u in $[a, \infty)$.

Proof. Assume that there were a real number a satisfying $U''(u) \leq 0$ for $a \leq u < \infty$. By Lemma 2.5, there exists a sequence $\{u_n\}$ in $\mathbf{R}$ satisfying

$$U'(u_n) \to 0 \text{ as } n \to \infty.$$

The tangent line of $U(u)$ at a point $(u_n, U(u_n))$ is

$$U(u) - U(u_n) = U'(u_n)(u - u_n).$$

Since $U''(u) \leq 0$ for $a \leq u < \infty$ from the assumption, we have

$$U(u+h) \leq U(u) = U(u_n) + U'(u_n)(u - u_n)$$

for non zero h with $a \leq u + h < \infty$. When $n \to \infty$, by Lemma 2.5, we have $U(u) \to 0$. Thus we have $U(u+h) \leq 0$ for $a \leq u + h < \infty$. This is a contradiction.

LEMMA 2.7. *Let $U(u)$ be a positive C^2 function with $U(u) \leq \exp(-u^2)$ for $-\infty < u < \infty$. Then we have either*

(i) *there exists a real number a such that $U''(u) \geq 0$ for all u in $[a, \infty)$, or*
(ii) *there exists a sequence $\{a_k\}$ of real numbers such that $U''(u) \geq 0$ for all u in $[a_{2k-2}, a_{2k-1}]$ and $U''(u) \leq 0$ for all u in $[a_{2k-1}, a_{2k}]$.*

Proof. By Lemma 2.5 and 2.6, we have the lemma.

Let $f(r,u)$ be a real valued C^2 function in $[0,1) \times (-\infty, \infty)$ and $\Phi(r\exp(\sqrt{-1}\theta), u + \sqrt{-1}v) = v - f(r,u)$ in $\Delta \times \mathbf{C}$.

LEMMA 2.8. *Let $R(r)$ and $U(u)$ be positive C^2 functions, respectively, satisfying the inequalities*

$$R(r) \leq \exp((\log r)^{-1})$$

and

$$U(u) \leq \exp(-u^2)$$

for $0 \leq r < 1$, $-\infty < u < \infty$ and let $\Phi(r,u,v) = v - R(r)U(u)$. If a domain

$$\Omega = \{r \exp(\sqrt{-1}\theta), u + \sqrt{-1}v) \in \mathbf{C}^2; \Phi(r,u,v) < 0, \quad 0 \leq r < 1, \ -\infty < u < \infty\}$$

is pseudoconvex and if there exists a real number a such that $U''(u) \geq 0$ for all u in $[a, \infty)$, then the Laplacian $\Delta_z R(r) \leq 0$ in $\{r \exp(\sqrt{-1}\theta) \in \mathbf{C}; 0 \leq r < 1\}$.

Proof. By Lemma 2.5 and 2.7, we have a sequence $\{u_n\}$ of real numbers satisfying

$$U''(u_n) \geq 0, \ U'(u_n) \to 0 \text{ as } n \to \infty.$$

Since the domain Ω is pseudoconvex, for $x = r \cos\theta$, $y = r \sin\theta$ and $\Phi = v - R(r)U(u)$, we have

$$\begin{aligned}
L(\Phi) = &\frac{1}{16}\{((\frac{\partial\Phi}{\partial u})^2 + (\frac{\partial\Phi}{\partial v})^2)\Delta_z\Phi + ((\frac{\partial\Phi}{\partial x})^2 + (\frac{\partial\Phi}{\partial y})^2)\Delta_w\Phi \\
&- 2(\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial u}\frac{\partial^2\Phi}{\partial x\partial u} + \frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial u}\frac{\partial^2\Phi}{\partial y\partial v} + \frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial v}\frac{\partial^2\Phi}{\partial x\partial v} \\
&- \frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial v}\frac{\partial^2\Phi}{\partial y\partial u} + \frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial u}\frac{\partial^2\Phi}{\partial y\partial u} - \frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial u}\frac{\partial^2\Phi}{\partial x\partial v} \\
&+ \frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial v}\frac{\partial^2\Phi}{\partial y\partial v} + \frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial v}\frac{\partial^2\Phi}{\partial x\partial u})\} \\
= &\frac{1}{16}\{2\frac{\partial(R(r)U(u))}{\partial r}\frac{\partial(R(r)U(u))}{\partial u}\frac{\partial^2(R(r)U(u))}{\partial r\partial u} \\
&- ((\frac{\partial(R(r)U(u))}{\partial u})^2 + 1)(\frac{\partial^2(R(r)U(u))}{\partial r^2} + \frac{1}{r}\frac{\partial(R(r)U(u))}{\partial r}) \\
&- (\frac{(R(r)U(u))}{\partial r})^2\frac{\partial^2(R(r)U(u))}{\partial u^2}\} \\
= &\frac{1}{16}\{2R(r)R'^2(r)U(u)U'^2(u) \\
&- (R^2(r)U'^2(u) + 1)(R''(r) + \frac{1}{r}R'(r))U(u) \\
&- R(r)R'^2(r)U^2(u)U''(u)\} \geq 0
\end{aligned}$$

for any $0 \leq r < 1$ and $-\infty < u < \infty$. For the pseudoconvex domain Ω and the sequence $\{u_n\}$, we have

$$\begin{aligned}
&R(r)R'^2(r)U^2(u_n)U''(u_n) + (R^2(r)U'^2(u_n) + 1)U(u_n)\Delta_z R(r) \\
\leq &2R(r)R'^2(r)U(u_n)U'^2(u_n)
\end{aligned}$$

for any $0 \leq r < 1$. Hence we have

$$\begin{aligned}
\Delta_z R(r) \leq &\frac{2R(r)R'^2(r)U'^2(u_n)}{R^2(r)U'^2(u_n) + 1} \\
&- \frac{R(r)R'^2(r)U(u_n)U''(u_n)}{R^2(r)U'^2(u_n) + 1}
\end{aligned}$$

for any $0 \leq r < 1$. Since the function $U(u_n)$ is positive and $U''(u_n) \leq 0$ for any $0 \leq r < 1$ and $n \geq 1$, and $R(r) > 0$, we have

$$\Delta_z R(r) \leq 2R(r)R'^2(r)U'^2(u_n).$$

Since $U'(u_n) \to 0$ as $n \to \infty$, we have

$$\Delta_z R(r) \leq 0$$

for any $0 \leq r < 1$.

LEMMA 2.9([20]). *There is no positive C^2 function $R(r)$ with $R(r) \leq \exp((\log r)^{-1})$ satisfying $\Delta_z R(r) \leq 0$ in $\{r \exp(\sqrt{-1}\theta) \in \mathbf{C}; 0 \leq r < 1\}$.*

THEOREM 2.10. *Let $U(u)$ be a positive C^2 function with $U(u) \leq \exp(-u^2)$ for $-\infty < u < \infty$. If there exists a real number a such that $U''(u) \leq 0$ for all u in $[a, \infty)$, then one cannot find a positive C^2 function $R(r)$ with $R(r) \leq \exp((\log r)^{-1})$ such that*

$$\Omega = \{(r \exp(\sqrt{-1}\theta), u + \sqrt{-1}v) \in \mathbf{C}^2; \Phi(r, u, v) < 0\}$$

is pseudoconvex, where $\Phi = v - R(r)U(u)$ for $0 \leq r < 1$ and $-\infty < u < \infty$.

Proof. Suppose taht $R(r)$ is a positive C^2 function with $R(r) \leq \exp((\log r)^{-1})$ for $0 \leq r < 1$, and satisfying the domain Ω is pseudoconvex. By Lemma 2.8, we have $\Delta_z R(r) \leq 0$ in $\{r \exp(\sqrt{-1}\theta) \in \mathbf{C}; 0 \leq r < 1\}$. This contradicts the statement of Lemma 2.9.

3. Stein neighborhood bases

A complex manifold Ω is a monotone union of polydiscs if $\Omega = \bigcup_{j=1}^{\infty} P_j$ where $P_1 \subset P_2 \subset \cdots$ and where each P_j is biholomorphically equivalent to a polydisc in $\mathbf{C}^n$, $\dim \Omega = n$. It is known that a monotone union of polydiscs need not be Stein. Here after, we exclusively suppose that complex manifolds are connected and paracompact. If Ω is a complex manifold with Kobayashi distance, then Ω is called a hyperbolic manifold. J. E. Fornaess-E. L. Stout [8] proved that if M is a monotone union of polydiscs in a taut complex manifold then M is biholomorphically equivalent to a polydisc, and proved that if the complex manifold Ω is a monotone union of polydiscs and hyperbolic then Ω is biholomorphically equivalent to a polydisc. Let a and b be real numbers with

$-\infty \leq a < b \leq \infty$ and $\theta(\Omega)$ be the set of all holomorphic functions on Ω. K. H. Shon [20] proved that if P^n is an open polydisc with multi-radius $(r_1, r_2, \cdots, r_n)$ and center 0 in $\mathbf{C}^n$ then there exists no Stein neighborhood bases of the product set $P^n \times (a,b)$ in $\mathbf{C}^n \times \mathbf{C}$.

LEMMA 3.1. *Let Ω be a complex manifold with $\dim \Omega = n$ and a monotone union of polydiscs containing 0 in $\mathbf{C}^n$. If Ω is hyperbolic, then there exists no Stein neighborhood bases of the product set $\Omega \times (a,b)$ in $\mathbf{C}^n \times \mathbf{C}$.*

Proof. If Ω is hyperbolic and $\Omega = \bigcup_{j=1}^{\infty} P_j$ which P_j is biholomorphically equivalent to a polydisc with center 0 in $\mathbf{C}^n$, then Ω is biholomorphically equivalent to a polydisc, by the result of [8]. Therefore, we may assume that the mapping $\Omega \to P^n$ is biholomorphically onto the open polydisc P^n in $\mathbf{C}^n$. Thus, from the result of [20], the product set $\Omega \times (a,b)$ has no Stein neighborhood bases in $\mathbf{C}^n \times \mathbf{C}$.

Let $\mathcal{U} = \{U_i; U_i \subset\subset P^n, i \in I\}$ be a locally finite Stein open covering of P^n, $z := (z_1, z_2, \ldots, z_n)$, $U_{ij} := U_i \cap U_j$ and $f_{ij}(z,t)$ be real valued functions in $U_{ij} \times (a,b)$ which are holomorphic in $z \in U_{ij}$ for all $i,j \in I$. Asystem $\{f_i\}_{i \in I}$ is called a solution for the Cousin distribution $\{f_{ij}\}_{i,j \in I}$ for $\mathcal{U}$ depending real analytically on a parameter $t \in (a,b)$ if there is a system of real analytic functions $\{f_i\}_{i \in I}$ on $U_i \times (a,b)$ such that f_i is holomorphic in $z \in U_i$ and $f_{ij} = f_j - f_i$ on $U_{ij} \times (a,b)$ for each $i,j \in I$. Let $\varphi_{P^n \times R}$ be the sheaf of the product set $P^n \times (a,b)$ of germs of real analytic functions, let N be the open neighborhood of the set $P^n \times (a,b)$ in $\mathbf{C}^n \times \mathbf{C}$

$$N = \{(z,w) \in \mathbf{C}^n \times \mathbf{C}; (z_1, z_2, \ldots, z_n) \in P^n,\ a < \operatorname{Re} w < b,$$
$$|\operatorname{Im} w| < \exp(\exp((\log \frac{|z_i|}{r_i})^{-1})) - 1,\ 1 \leq i < n\},$$

and set $C^{\infty}_{(0,1)}(N)$ be the set of C^{∞} forms of type $(0,1)$ on N. We investigate the Cauchy-Riemann equation $\frac{\partial \mu}{\partial \bar{z}} = \nu$ on $\Omega \times (a,b)$. The main methods are based on the result of H. Kazama [11]. By Lemma 4 of [11], if D be a connected and simply connected open neighborhood of $\Delta \times (a,b)$ in $\mathbf{C} \times \mathbf{C}$ and $D(z) = \{w \in \mathbf{C}; (z,w) \in D\}$ for $z \in \Delta$ then $\Delta \times \bigcup_{z \in \Delta} D(z)$ is the envelope of holomorphy of D.

LEMMA 3.2. *Under the assumption of Lemma 3.1, there are an open neighborhood G of $\Omega \times (a, b)$ in $\mathbf{C}^n \times \mathbf{C}$ and a $\overline{\partial}$-closed $f \in C^\infty_{(0,1)}(G)$ such that for any open neighborhood H with $\Omega \times (a,b) \subset H \subset G$ the restriction $f|_H$ is not $\overline{\partial}$-exact on H.*

Proof. By the assumption, we may assume that the open set Ω is biholomorphically equivalent to the unit open polydisc $P^n(1)$. We suppose that $a + b = 0$ and $b > 1$. We take an open neighborhood of $P^n(1) \times (-b, b)$

$$G = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}; (z_1, z_2, \cdots, z_n) \in P^n(1),\ |\mathrm{Re}\, w| < b,\\ |\mathrm{Im}\, w| < \exp(\exp((\log |z_i|)^{-1})) - 1,\ 1 \leq i \leq n\}$$

and, we put

$$G(m) = \{w \in \mathbf{C}; |\mathrm{Re}\, w| < b,\ |\mathrm{Im}\, w| < \exp(\exp((\log \frac{m}{m+1})^{-1})) - 1\}$$

for $m = 1, 2, \cdots$. Let $g_m(w)$ be a holomorphic function on $G(m)$ which is not holomorphically extendible at the boundary of $G(m)$ and p be the projection $(z, w) \to w$ from $P^n(1) \times (-b, b)$ into $(-b, b)$. If we let closed sets $F(m)$ in G

$$F(m) = \{(z, w) \in G; p(z, w) \notin G(m)\}\\ \cup \{(z, w) \in G; |z_i - \frac{m}{m+1}| \geq \frac{1}{2(m+1)(m+2)},\ 1 \leq i \leq n\}$$

and let an n-tuple $[\frac{m}{m+1}] = (\frac{m}{m+1}, \cdots, \frac{m}{m+1})$, then we have

$$F(m) \cap ([\frac{m}{m+1}] \times G(m)) = \emptyset$$

for $m = 1, 2, \cdots$. Hence we have a C^∞ function $\phi_m : G \to [0, 1]$ such that

$$\phi_m(z, w) = \begin{cases} 0 & \text{in a neighborhood of } F(m) \text{ in } G \\ 1 & \text{in a neighborhood of } [\frac{m}{m+1}] \times G(m) \text{ in } G \end{cases}$$

for each $m = 1, 2, \cdots$. If we put

$$f(z,w) = \sum_{m=1}^{\infty}(z_1 - \frac{m}{m+1})^{-1}(z_2 - \frac{m}{m+1})^{-1}\cdots (z_n - \frac{m}{m+1})^{-1} g_m(w)\overline{\partial}\phi_m(z,w) = \sum_{m=1}^{\infty}(z - \frac{m}{m+1})^{-1} g_m(w)\overline{\partial}\phi_m(z,w),$$

then f is a $(0,1)$ type of class C^∞ in G. Since $g_m(w)$ is holomorphic on $G(m)$, we have $\overline{\partial} f(z,w) = 0$. Assume that there a C^∞ function h on any connected open neighborhood H of $P^n(1) \times (-b,b)$ in G satisfying $f|_H = \overline{\partial} h$. We set

$$\zeta_m(z) = (z - \frac{m}{m+1})^{-1}(1 - \exp 2\pi\sqrt{-1}(m+1)z).$$

Then

$$\eta(z,w) := \sum_{m+1}^{\infty}(\zeta_m(z) g_m(w)\phi_m(z,w) - (1 - \exp 2\pi\sqrt{-1}(m+1)z)h(z,w))$$

is a C^∞ function and

$$\overline{\partial}\eta(z,w) = (1 - \exp 2\pi\sqrt{-1}(m+1)z)\cdot (\sum_{m+1}^{\infty}(z - \frac{m}{m+1})^{-1} g_m(w)\overline{\partial}\phi_m(z,w) - \overline{\partial} h(z,w)) = (1 - \exp 2\pi\sqrt{-1}(m+1)z)(f(z,w) - \overline{\partial} h(z,w)) = 0$$

in $(z,w) \in H$, that is, $\eta(z,w) \in \theta(H)$. Since we have

$$\lim_{z\to[\frac{m}{m+1}]} \zeta_m(z) = (-2\pi\sqrt{-1}(m+1)\exp 2\pi\sqrt{-1}(m+1)z)\big|_{z=[\frac{m}{m+1}]} = -2\pi\sqrt{-1}(m+1)$$

and

$$\phi_k([\frac{m}{m+1}], w) = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n, \end{cases}$$

we have

$$\eta([\frac{m}{m+1}], w) = -2\pi\sqrt{-1}(m+1)g_m(w).$$

If we take a connected and simply connected open neighborhood $\hat{H}$ of $P^n(1) \times (-b, b)$ in $\mathbf{C}^n \times \mathbf{C}$ with $\hat{H} \subset H$, then the product set $P^n(1) \times \bigcup_{z \in P^n(1)} \hat{H}(z)$ is the envelope of holomorphy of $\hat{H}$ where $\hat{H}(z) = \{w; (z, w) \in \hat{H}\}$, and hence we have a function $\tilde{\eta} \in \theta(P^n(1) \times \bigcup_{z \in P^n(1)} \hat{H}(z))$ such that $\tilde{\eta}|_H = \eta$. Since the difference $\bigcup_{z \in P^n(1)} \hat{H}(z) - G(m) \neq \phi$ for large m and $\tilde{\eta}([\frac{m}{m+1}], w)$ is holomorphic in $w \in \bigcup_{z \in P^n(1)} \hat{H}(z)$ for $m = 1, 2, \cdots$, this contradicts the assumption.

LEMMA 3.3. *Under the assumption of Lemma 3.2, if $\mathcal{U} = \{U_i; U_i \subset\subset \Omega,\ i \in I\}$ is a locally finite Stein open covering of Ω, then there exists a Cousin distribution $\{f_{ij}\}_{i,j \in I}$ for $\mathcal{U}$ depending real analytically on the parameter $t \in (a, b)$ which has no solution.*

Proof. Since Ω is hyperbolic, the covering $\mathcal{U}$ is hyperbolic. We prove the theorem for $\mathcal{U} = \{U_i; U_i \subset\subset P^n(1),\ i \in I\}$ and $t \in (-b, b)$. For the norm $|z| = \max_{1 \leq \nu \leq n} |z_\nu|$ on $\mathbf{C}^n$, we put $\delta_i = \sup\{|z|; z \in U_i\}$ for $U_i \subset\subset P^n(1)$ and

$$G_i = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}; z \in U_i,\ |\mathrm{Re}\, w| < b, \\ |\mathrm{Im}\, w| < \exp(\exp((\log \delta_i)^{-1})) - 1\}.$$

Then the set G_i is a Stein open subset of G of Lemma 3.2. Hence we have a C^∞ function ζ_i on G_i such that $\overline{\partial}\zeta_i = f$ for $\overline{\partial}$-closed $f \in C^\infty_{(0,1)}(G)$ and each i. We put $\zeta_{ij} = \zeta_j - \zeta_i$ on $G_i \cap G_j$ and $f_{ij} = \zeta_{ij}|_{U_{ij} \times (-b,b)}$. Then $\{f_{ij}\}_{i,j \in I}$ is the Cousin distribution for $\mathcal{U}$ depending real analytically on the parameter $t \in (-b, b)$. Assume that there were a solution $\{f_i\}_{i \in I}$

for the Cousin distribution $\{f_{ij}\}_{i,j\in I}$. Since $f_i(z,t)$ is real analytic on $U_i \times (-b,b)$ and holomorphic in $z \in U_i$, there is a holomorphic function $h_i(z,w)$ in an open neighborhood H_i of $U_i \times (-b,b)$ in $P^n(1) \times (-b,b)$ such that $h_i|_{U_i\times(-b,b)} = f_i$. There exists an open subset of H of G satisfying

$$P^n(1) \times (-b,b) \subset H \subset (\bigcup_{i\in I} G_i) \cap (\bigcup_{i\in I} H_i)$$

and $\zeta_{ij} = h_j - h_i$ on $H_i \cap H_j \cap H$. Therefore, we have $\zeta := \zeta_i - h_i \in C^\infty(H)$ and $f|_H = \overline{\partial}\zeta$. This contradicts the statement of Lemma 3.2.

THEOREM 3.4. *Under the assumption of Lemma 3.3, there is a real analytic function $g(z,t)$ in $\Omega \times (a,b)$ such that one cannot find a real analytic function $f(z,t)$ in $\Omega \times (a,b)$ satisfying $\frac{\partial f(z,t)}{\partial \overline{z}} = g(z,t)$ in $\Omega \times (a,b)$.*

Proof. Let $\{f_{ij}\}_{i,j\in I}$ be the Cousin distribution for $\mathcal{U}$ depending real analytically on the parameter $t \in (-b,b)$ as in the proof of Lemma 3.3. By H. Grauert [9] and B. Malgrange [13], we have

$$\begin{aligned} &H^1(P^n(1)\times(-b,b), \varphi_{P^n(1)\times(-b,b)}) \\ &\qquad = H^1(\{U_i \times (-b,b)\}_{i\in I}, \varphi_{U_i\times(-b,b)}) = 0. \end{aligned}$$

Hence, we have a system $\{g_i \in H^0(U_i\times(-b,b),\ \varphi_{U_i\times(-b,b)})\}_{i\in I}$ satisfying $f_{ij} = g_j - g_i$ on $U_{ij} \times (-b,b)$, and

$$\frac{\partial f_{ij}}{\partial \overline{z}} = \frac{\partial g_j}{\partial \overline{z}} - \frac{\partial g_i}{\partial \overline{z}} = 0$$

on $U_{ij} \times (-b,b)$. We can write $g = \frac{\partial g_i}{\partial \overline{z}}$ in $P^n(1) \times (-b,b)$ for suitable $g_i \in H^0(U_i\times(-b,b),\ \varphi_{U_i\times(-b,b)})$. Assume that there were a real analytic function f in $P^n(1) \times (-b,b)$ satisfying $\frac{\partial f}{\partial \overline{z}} = g$ for the real analytic function g. Then we have $\frac{\partial g_i}{\partial \overline{z}} - \frac{\partial f}{\partial \overline{z}} = 0$ in $U_i \times (-b,b)$. If we put $f_i := g_i - f$ in $U_i \times (-b,b)$, then f_i are holomorphic in $z \in U_i$ for all $i \in I$ and $f_j - f_i = g_j - g_i = f_{ij}$. That is, $\{f_i\}_{i\in I}$ is a solution for the Cousin distribution $\{f_{ij}\}_{i,j\in I}$. This is a contradiction.

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