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LANGLANDS PROGRAM

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1. Introduction

This article is a continuation of the previous one [**7**] about abelian class field theory. In algebraic number theory, a fundamental problem is to describe how an ordinary prime p factors into primes in the ring of integers of an arbitrary finite extension F of $\mathbf{Q}$ and generalize it to an arbitrary number field over F.

Unless otherwise specified, E is Galois over F with Galois group $G = G(E/F)$. Let $\mathfrak{p}$ be a (finite) prime of F and $\mathfrak{P}_i$ primes of E lying over $\mathfrak{p}$. Then

$$(*) \qquad \mathfrak{p}\mathcal{O}_F = (\mathfrak{P}_1, \dots, \mathfrak{P}_g)^e.$$

Recall that $efg = n$ with $n = [E : F]$. We define the *decomposition group* $G(\mathfrak{P}) = \{\sigma \in G \mid \sigma\mathfrak{P} = \mathfrak{P}\}$, which is a finite group of order ef. Consider a homomorphism from $G(\mathfrak{P})$ onto $G(\overline{\mathcal{O}}_E/\overline{\mathcal{O}}_F)$ defined by $\bar{\sigma}(x \bmod \mathfrak{P}) = \sigma(x) \bmod \mathfrak{P}$ for $\sigma \in G(\mathfrak{P})$. Then its kernel is the *inertia group* $T(\mathfrak{P}) = \{\sigma \in G(\mathfrak{P}) \mid \sigma x \equiv x \bmod \mathfrak{P}$ for all $x \in \mathcal{O}_E\}$ whose order is e. As is well known, $G(\mathfrak{P})/T(\mathfrak{P}) \cong G(\overline{\mathcal{O}}_E/\overline{\mathcal{O}}_F)$ which is cyclic of order f. Its canonical generator is called the *Frobenius automorphism*, which we denote by $Fr_{\mathfrak{P}}$. To be sure, $Fr_{\mathfrak{P}}$ is an automorphism of E over F determined only up to conjugacy in G. Nevertheless the resulting conjugacy class $\{Fr_{\mathfrak{P}}\}$ completely determines the factorization type of $(*)$. For example, $\mathfrak{p}$ splits completely in E if and only if $\{Fr_{\mathfrak{P}}\}$ is the class of the identity alone. Therefore, to obtain information about the factorization of such $\mathfrak{p}$, attention is focused on the Frobenius automorphism of G.

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Hecke and Artin L-function Let χ be a Hecke character of F (i.e., a character of the ray class group C mod $\mathfrak{m}$ ([7], §4)). We define the *Hecke L-function* by

$$L(s,\chi)=\sum_{\mathfrak{A}}\frac{\chi(\mathfrak{A})}{N\mathfrak{A}^s}\,,\qquad \mathrm{Re}(s)>1$$

the sum extending over all nonzero integral ideals. Then it has an Euler product

$$L(s,\chi)=\prod_{\mathfrak{p}}(1-N\mathfrak{p}^{-s}\chi(\mathfrak{p}))^{-1}$$

and has analytic continuation to $\mathbf{C}$ with functional equation $L(s,\chi)=\varepsilon(s,\chi)L(1-s,\chi^{-1})$. If χ is the trivial character χ_0, then $L(s,\chi_0)$ is nothing but a Dedekind zeta function $\zeta_F(s)$ of F which has a simple pole at $s=1$. Otherwise, $L(s,\chi)$ is entire. In case that $F=\mathbf{Q}$ and hence we take a modulus $\mathfrak{m}$ of the form $(m)P_\infty$, $\chi\in\widehat{(\mathbf{Z}/m\mathbf{Z})^*}$ i.e., χ is a Dirichlet character mod m. Therefore $L(s,\chi)$ becomes a Dirichlet L-function $\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$.

Let V be an n-dimensional complex vector space and let $\sigma:G\to GL(V)$ be a representation of G in V. Then $\sigma(Fr_{\mathfrak{P}})$ belongs to $\mathrm{End}\,(V^{T(\mathfrak{P})})$. Let I be an $n\times n$ identity matrix and set

$$L_{\mathfrak{p}}(s,\sigma)=\det(I-N\mathfrak{p}^{-s}\sigma(F_{\mathfrak{P}}))^{-1}.$$

Since almost all $\mathfrak{p}$ are unramified in E, $V^{T(\mathfrak{P})}=V$ and so the Euler factor $L_{\mathfrak{p}}(s,\sigma)$ is of degree n in $N\mathfrak{p}^{-s}$. The *Artin L-function* is defined by

$$L(s,\sigma)=\prod_{\mathfrak{p}}L_{\mathfrak{p}}(s,\sigma)\,,\qquad \mathrm{Re}(s)>1$$

where $\mathfrak{p}$ runs over all unramified primes.

THEOREM 1 (*Artin's reciprocity*). *Suppose E is abelian over F, and $\sigma:G\to\mathbf{C}^*$ is a character. Then there exists a Hecke character χ of F such that $L(s,\sigma)=L(s,\chi)$ (Hecke's abelian L-function for F).*

When $F=\mathbf{Q}$, one seeks to describe $\{Fr_p\}$ (and hence the factorization of p in E) intrinsically in terms of p and the arithmetic of $\mathbf{Q}$. To see what this means, consider the example $E = \mathbf{Q}(\sqrt{-1})$ with $\mathcal{O}_E = \mathbf{Z}(\sqrt{-1})$. In this case, $G = G(E/\mathbf{Q})$ can be identified with the subgroup $\{\pm 1\}$ of $\mathbf{C}^*$ via the obvious isomorphism $\sigma : G \to \{\pm 1\}$. Then we have

$$\sigma(Fr_p) = \left(\frac{-1}{p}\right)$$

with $\left(\frac{-1}{p}\right)$ the Legendre symbol. To express it in terms of a congruence condition on p instead of on -1, we appeal to a part of the quadratic reciprocity law for $\mathbf{Q}$ which states (for odd p, precisely those p unramified in E)

$$\left(\frac{-1}{p}\right) = (-1)^{p-1/2}, \quad \text{i.e.,} \quad \sigma(Fr_p) = (-1)^{p-1/2}.$$

This is the type of intrinsic description of Fr_p we sought ; from it and the fact that

$$(-1)^{p-1/2} = 1 \quad \text{if and only if} \quad p \equiv 1 \bmod 4,$$

we conclude that the factorization of p in $\mathbf{Z}(\sqrt{-1})$ depends only on its residue modulo 4. In particular, all primes in a given arithmetic progression mod4 have the same factorization type in $\mathbf{Z}(\sqrt{-1})$.

A major goal of class field theory is to give a similar description of $\{Fr_p\}$ for arbitrary Galois extensions E. However, this goal is far from achieved. In general, we can not expect there to be a modulus N such that $\{Fr_p\} = \{1\}$ if and only if p lies in some arithmetic progression mod$N$. However, if E is abelian, then a great deal can be said. Indeed, suppose E is such an extension and $\sigma : G \to \mathbf{C}^*$ is a character. By Artin's reciprocity, there exist a positive integer N and a Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$ such that $\sigma(Fr_p) = \chi(p)$ for all unramified primes p.

The question remains : for nonabelian Galois extensions, how can the family $\{Fr_p\}$ be described in terms of the ground field $\mathbf{Q}$?

Recognizing the use of groups in term of their matrix representations, Artin focused attention on n-dimensional representations of the Galois group G. In this way he was able to transfer the problem analyzing certain conjugacy classes in G to an analogous problem inside $GL_n(\mathbf{C})$.

For arbitrary E and σ, Artin was able to derive important analytic properties of $L(s,\sigma)$. However, he was unable to discover the appropriate n-dimensional analogues of Dirichlet's characters and L-functions. Although some such 2-dimensional automorphic L-functions were being studied nearly by Hecke, it remained for Langlands (40 years later) to see the connection and map out some general conjectures. He isolated the notion of an automorphic representation of the group GL_n over the adeles of $\mathbf{Q}$ as the appropriate generalization of a Dirichlet character. Furthermore, he associated L-functions with these automorphic representations, generalizing Dirichlet's L-functions in the case $n = 1$. Finally he conjectured that each n-dimensional Artin L-function $L(s,\sigma)$ is exactly the automorphic L-function $L(s,\pi_\sigma)$ for some automorphic representation π_σ of GL_n. The conjectured correspondence $\sigma \to \pi_\sigma$ is to be regarded as for reaching generalization of Artin's reciprocity map $\sigma \to \chi_\sigma$. In case $n = 2$, when π_σ corresponds to a classical automorphic form $f(z)$ in the sense of Hecke([**11**], [**12**]), the map $\sigma \to \pi_\sigma$ affords an interpretation of the classes $\{Fr_p\}$ in terms of certain conjugacy classes in $GL_2(\mathbf{C})$ determined by the Fourier coefficients of the form $f(z)$.

2. Automorphic Representations

Let F be a number field and $G = GL_n(F)$ as a reductive algebraic group over F. For each place v, let $G_v = GL_n(F_v)$ and $G_{\mathcal{O}_v} = GL_n(\mathcal{O}_v)$. Since $G_{\mathcal{O}_v}$ is a maximal compact open subgroup of G_v, we define the *adele group* $G_{\mathbf{A}}$ of G by the restricted direct product

$$G_{\mathbf{A}} = \prod_v (G_v : G_{\mathcal{O}_v}),$$

which we also write $G_{\mathbf{A}} = GL_n(\mathbf{A}_F)$. Then the center of $G_{\mathbf{A}}$ is

$$Z_{\mathbf{A}} = \left\{ \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix} \middle| a \in \mathbf{A}_F^* \right\}.$$

Now we consider the Hilbert space $L_0^2 = L_0^2(Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}})$ of square-integrable functions ϕ on $Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}$ satisfying a certain cuspidal condition. By G_F we denote the group G regarded as a discrete subgroup of $G_{\mathbf{A}}$ via the diagonal embedding $\alpha \to (\alpha, \ldots, \alpha, \ldots)$. In harmonic analysis as a chapter in the theory of group representation ([**5**], [**10**]), as well as in the theory of automorphic forms, the fundamental example of a unitary representation is the so-called right regular representation of $G_{\mathbf{A}}$. Let R_0 denote the unitary representation of $G_{\mathbf{A}}$ defined in L_0^2 by the formula $R_0(g)\phi(h) = \phi(hg)$. This is the *right regular representation* of $G_{\mathbf{A}}$ in $L_0^2(Z_{\mathbf{A}}G_F \backslash G_{\mathbf{A}})$ induced from the trivial representation of $Z_{\mathbf{A}}G_F$.

Let π be an irreducible unitary representation of $G_{\mathbf{A}}$ in some space H_π. It is known ([**3**]) that π can be factored as a restricted tensor product $\pi = \bigotimes_v \pi_v$ with each π_v an irreducible unitary representation of G_v. This is analogous to the factorization $\prod_v \chi_v$ for Hecke characters. By a *restricted tensor product* $\bigotimes_v \pi_v$ we mean that its space H_π consists of vectors $w = \bigotimes_v w_v$ for which w_v is $G_{\mathcal{O}_v}$-fixed for almost all v. In particular, for almost all v, π_v is *unramified* ; this means that its space H_v contains vectors which are invariant under $G_{\mathcal{O}_v}$-action. Thus the representation $\pi(g)$ is well defined by the formula

$$\pi(g)w = \prod_v \pi_v(g_v)w_v$$

because $\pi_v(g_v)w_v$ belongs to $H_v^{G_{\mathcal{O}_v}}$ for almost all v. We say that π *appears* in R_0 (or is *equivalent to some subrepresentation of* R_0) if there exists an isomorphism A from H_π onto a subspace V_π of L_0^2 such that $A\pi(g) = R_0(g)A$ for all g in $G_{\mathbf{A}}$. An irreducible unitary representation π of $G_{\mathbf{A}}$ is called an *automorphic* (*cuspidal*) *representation* of G if it appears in the right regular representation R_0 of $G_{\mathbf{A}}$.

Examples of Automorphic Representations (1) An automorphic representation of $GL_1(F)$ is just a Hecke character $\chi = \prod_v \chi_v$ on the idele class group $\mathbf{A}_F^*/F^*$.

(2) A classical cusp form f of weight k for $SL_2(\mathbf{Z})$ which is also an eigen form for all the Hecke operators (i.e., $T(p)f = a_p f$ for all p) uniquely determines an automorphic representation $\pi_f = \bigotimes_p \pi_p$ of $GL_2(\mathbf{Q})$. Conversely, every automorphic representation $\pi = \bigotimes_p \pi_p$ determines some classical cusp form $f(z)$ (, not necessarily for $SL_2(\mathbf{Z})$). For f as above, the p-th Euler factor of its Dirichlet series ([**11**], [**12**]) $\phi_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$

is

$$L_p(s) = (1 - a_p p^{-s} + p^{k-1-2s})^{-1} = ((1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}))^{-1}.$$

3. Automorphic L-functions

Fix a finite place v of F, and suppose π_v is unramified representation of G_v. Then the theory of spherical functions assigns to π_v an element A_v of the complex torus T^n which is unique up to the natural action of the permutation group S_n on T^n. The procedure is this : let H_v^0 be the space of $G_{\mathcal{O}_v}$-fixed vectors in the space of π_v, and let $\mathcal{H}_v(G_v, G_{\mathcal{O}_v})$ denote the Hecke algebra over $\mathbf{C}$ (i.e., the convolution algebra of locally constant, $G_{\mathcal{O}_v}$-bi-invariant and compactly supported $\mathbf{C}$-valued function on G_v). Then the formula

$$\pi_v(f)w_v = \int_{G_v} f(g)\pi_v(g)w_v\, dg = \chi_{\pi_v}(f)w_v$$

defines a representation of $\mathcal{H}_v(G_v, G_{\mathcal{O}_v})$ in the one-dimensional space H_v^0. Therefore the resulting character $f \to \chi_{\pi_v}(f)$ defines an element A_v of T^n via the Satake isomorphism ([**1**], [**2**]), which is unique up to permutation. In fact, π_v is itself determined by the class of A_v. Thus there is an injection $\pi_v \mapsto A_v$ taking unramified representations of G_v to semisimple conjugacy classes in the group $GL_n(\mathbf{C})$.

Turning to the global situation, given $\pi = \otimes \pi_v$ let S_π be the finite set of places v outside of which π_v is unramified. Consider the family of conjugacy classes $\{A_v\}$, $v \notin S_\pi$. The point is that this construction generalizes the assignment of conjugacy classes which is implicit in the classical theory of Hecke, and yet it makes sense for all (not necessary automorphic) representations π of $GL_n(\mathbf{A}_F)$.

For instance, if $n = 1$ and π_v is an unramified character χ_v of F_v^*, then A_v is simply $\chi_v(\tilde{w}_v)$, the value of χ_v at any local uniformizing parameter. Furthermore if $\chi = \prod \chi_v$ is automorphic, that is, trivial on F^*, then the family $\{A_v\}$, $v \notin S_\pi$, actually determines χ uniquely.

Next suppose $n = 2$ and $\pi_f = \otimes \pi_p$ is an automorphic representation of GL_2 corresponding to the classical cusp form $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ in $S_k(SL_2(\mathbf{Z}))$. If $T(p)f = a_p f$ for all p, then

$$A_p = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix} \quad \text{in} \quad GL_2(\mathbf{C})$$

with $\alpha_p \beta_p = p^{k-1}$ and $\alpha_p + \beta_p = a_p$. In particular, the eigenvalues a_p completely determine the local representations π_p, and the family of classes $\{A_p\}$ completely determines π_f (or f).

Langlands' theory begins by attaching an L-function to any irreducible unitary representation $\pi = \otimes \pi_v$ of $G_{\mathbf{A}}$. Given π, let S_π be the finite set described above. For each $v \notin S_\pi$, let A_v denote the semisimple conjugacy class

$$A_v = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{pmatrix}$$

in $GL_n(\mathbf{C})$ corresponding to π_v. We consider the Euler product

$$L(s,\pi) = \prod_{v \notin S_\pi} L(s,\pi_v)$$

with $L(s,\pi_v) = \det(I - Nv^{-s}A_v)^{-1}$. This is an Euler product of degree n in the sense that each Euler factor $L(s,\pi_s)$ is of the form $P^{-1}(Nv^{-s})$ with P a polynomial of degree n and $P(0) = 1$. Then it can be shown that the infinite product converges for sufficiently large $\mathrm{Re}(s)$.

THEOREM 2. ([**4**], [**6**]) *Suppose $\pi = \otimes \pi_v$ is an arbitrary irreducible representation of $G_{\mathbf{A}}$. Then for all v one can define Euler factors $L(s,\pi_v)$ of degree $\leq n$, and local factors $\varepsilon(s,\pi_v)$ such that $\varepsilon(s,\pi_v)$ is 1 for almost all v, and*

$$L(s,\pi_v) = \det(I - Nv^{-s}A_v)^{-1}$$

whenever v is unramified. Moreover, if π is an automorphic representation, then the Euler product

$$L(s,\pi) = \prod_v L(s,\pi_v)$$

, initially defined only in some right half-plane, satisfies the following properties :

(i) *it extends to an entire function of $\mathbf{C}$ unless $n = 1$ and π is the trivial character, in which case $L(s,\pi)$ has a pole*

(ii) *it satisfies the functional equation*

$$L(s,\pi) = \varepsilon(s,\pi)L(1-s,\tilde{\pi})$$

with $\tilde{\pi}$ *the representation contragradient to* π, *and*

$$\varepsilon(s,\pi) = \prod_v \varepsilon(s,\pi_v).$$

The Euler product $L(s,\pi)$ is called an *automorphic L-function* if π is an automorphic representation. When $n=1$, π becomes a Hecke character $\chi = \prod_v \chi_v$ and A_v is simply $\chi_v(\tilde{\omega}_v)$ and so the theorem reduces to the work of Tate ([**13**]). In case $n=2$ it represents a vast reformulation and generalization of Hecke's work. In particular, it sheds light on the Euler product expansion of Hecke's Dirichlet series

$$\phi_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = (2\pi)^s \Gamma(s)^{-1} L(s,\pi_f)$$

associated with a normalized cusp form f of $S_k(SL_2(\mathbf{Z}))$ which is an eigen form for all the Hecke operators $T(p)$. It also interprets the constant in the functional equation of ϕ_f in terms of the local groups G_v. This last point underscores one of the key contributions of the original work of Jacquet and Langlands.

Now we are ready to state *Langlands' reciprocity conjecture.*

CONJECTURE 1 (*Langlands*): Suppose E is a finite Galois extension of F with Galois group $G = G(E/F)$, and $\sigma : G \to GL_n(\mathbf{C})$ is an irreducible representation of G. Then there is an automorphic representation π_σ of $GL_n(F)$ such that $L(s,\sigma) = L(s,\pi_\sigma)$.

If $n=1$ and E over F is abelian, it reduces to Artin's reciprocity law (Th. 1).

CONJECTURE (*Artin*): Suppose $\dim(\sigma) > 1$ and σ is irreducible. Then the Artin L-function $L(s,\sigma)$ is actually entire.

For $n = \dim(\sigma) > 1$, the truth of the conjecture 1 implies Artins conjecture on the entirety of his $L(s,\sigma)$ because the automorphic L-function $L(s,\pi_\sigma)$ is entire by Th. 2.

4. Generalized L-functions

Suppose G is an arbitrary reductive algebraic group defined over F, and E is a (sufficiently large) Galois extension of F. We recall from the theory of algebraic groups that there is a one-to-one correspondence between isomorphism classes of reductive algebraic groups G over the algebraic closure $\overline{F}$ and isomorphism classes of root systems $\Phi(G)$. Using maximal tori and root systems, one can define an *L-group* ${}^LG^0$ of G as a complex reductive Lie group ([**8**]). We consider an action of $G(E/F)$ on ${}^LG^0$ to define the *Galois form of L-group of G* as the semidirect product ${}^LG = {}^LG^0 \rtimes G(E/F)$. If G is split over F (in particular, when $G = GL_n$), ${}^LG$ reduces to a direct product because then the action of $G(E/F)$ on ${}^LG^0$ is trivial. In general, if v is a prime of F unramified in E with corresponding Frobenius automorphism Fr_v, there exists a one-to-one correspondence between unramified representations π_v of G_v and conjugacy classes $t(\pi_v)$ in ${}^LG$ such that the projection of $t(\pi_v)$ onto $G(E/F)$ is the class of Fr_v ([**1**]). By a *representation* of ${}^LG$ we mean a homomorphism $r : {}^LG \to GL_n(\mathbf{C})$ whose restriction to ${}^LG^0$ is complex analytic. By an *L-homomorphism* of Galois form of L-groups ${}^LG$ and ${}^LG'$ we understand a continuous homomorphism $\rho : {}^LG \to {}^LG'$ which is compatible with the natural projections of each group onto $G(E/F)$ and whose restriction to ${}^LG^0$ is a complex analytic map ${}^LG^0$ to ${}^LG'^0$.

In terms of these concepts we can state the following two conjectures.

CONJECTURE 2 (*Langlands*): Suppose $\pi = \otimes\pi_v$ is an automorphic representation of G and r is a finite-dimensional representation of ${}^LG$. Then the Euler product (or L-function)

$$L(s,\pi,r) = \prod_{\substack{v \\ \text{unramified}}} \det(I - r(t(\pi_v))Nv^{-s})^{-1}$$

, initially defined in a right-half plane of s, has analytic continuation to $\mathbf{C}$ with functional equation relating $L(s,\pi,r)$ to $L(1-s,\tilde{\pi},r)$.

CONJECTURE 3 (*Langlands' Functoriality*): Suppose G and G' are reductive algebraic groups and $\rho : {}^LG \to {}^LG'$ is an L-homomorphism. Then to each automorphic representation $\pi = \otimes\pi_v$ of G there exists an automorphic representation $\pi' = \otimes\pi'_v$ of G' such that $S_\pi = S_{\pi'}$ and for all $v \notin S_{\pi'}$, $t(\pi'_v)$ is the conjugacy class in ${}^LG'$ which contains $\rho(t(\pi_v))$. Moreover, for any finite-dimensional representation r' of ${}^LG'$,

$$L(s,\pi',r') = L(s,\pi,r'\circ\rho).$$

5. Concluding Remarks

Langlands' Functoriality does imply all the preceding conjectures discussed.

First take $G' = GL_n(F)$, and suppose r is any n-dimensional representation of ${}^LG$ (G arbitrary but fixed). Since G' is a reductive split algebraic group, ${}^LG'^0 = GL_n(\mathbf{C})$ ([8]). Thus, let $\rho : {}^LG \to {}^LG'$ be an L-homomorphism such that the following diagram commutes :

$$\begin{array}{ccc} {}^LG & \xrightarrow{\quad r \quad} & GL_n(\mathbf{C}) \\ \rho \searrow & & \nearrow r' = St \\ & {}^LG' = GL_n(\mathbf{C}) \times G(E/F) & \end{array}$$

where $St : {}^LG' \to GL_n(\mathbf{C})$ is the standard representation of ${}^LG'$, namely projection onto the L-group ${}^LG'^0 = GL_n(\mathbf{C})$. If Conjecture (3) is true, then we have a lift of automorphic representations $\pi \to \pi'$ between G and G' with

$$L(s,\pi,r) = L(s,\pi,St\circ\rho) = L(s,\pi',St) = L(s,\pi').$$

The last equality is due to $St(t(\pi'_v)) = A'_v$. It follows from Th. 2 that Conjecture (3) implies Conjecture (2).

Next, suppose $G = \{e\}$ (the trivial group) and again take $G' = GL_n(F)$. Let σ be an irreducible representation from $G(E/F)$ into $GL_n(\mathbf{C})$. Then the only possible automorphic representation of G is 1 (the trivial one) and an L-homomorphism $\rho_\sigma : {}^LG \to {}^LG'$ amounts to specifying σ such that $\rho_\sigma(1,\gamma) = (\sigma(\gamma),\gamma)$ for all $\gamma \in G(E/F)$, i.e., we have the following commutative diagram

$$\begin{array}{cccc} (1,\gamma) & {}^LG = \{1\} \times G(E/F) & \xrightarrow{\quad r \quad} & GL_n(\mathbf{C}) \\ \Big\downarrow & \Big\downarrow{\scriptstyle \rho=\rho_\sigma} & & \nearrow r' = St \\ (\sigma(\gamma),\gamma) & {}^LG' = GL_n(\mathbf{C}) \times G(E/F) & & \end{array}$$

If Conjecture (3) is true, there is an automorphic representation $\pi'_\sigma = \otimes\pi'_v$ of G' (associated to the trivial automorphic representation of G via $\rho = \rho_\sigma$, i.e., $\rho_\sigma(t(1)) \subset t(\pi'_v)$) such that for any unramified prime v, the projection of $t(\pi'_v)$ on $GL_n(\mathbf{C})$ is just $\sigma(Fr_v)$. Moreover,

$$L(s,1,r) = L(s,1,r' \circ \rho) = L(s,1,St \circ \rho) = L(s,\pi'_\sigma,St) = L(s,\pi'_\sigma).$$

On the other hand, due to the fact that $r(t(1)) = St \circ \rho_\sigma(t(1)) = \sigma(Fr_v)$,

$$\begin{aligned} L(s,1,r) &= \prod_{\substack{v \\ \text{unramified}}} \det(I - Nv^{-s} r(t(1))) \\ &= \prod_{\substack{v \\ \text{unramified}}} \det(I - Nv^{-s} \sigma(Fr_v)) \\ &= L(s,\sigma), \end{aligned}$$

which is just an Artins L-function.

Therefore Conjecture (3) implies Conjecture (1).

Furthermore, the truth of Conjecture (3) automatically implies Artin's conjecture on his $L(s,\sigma)$. We note that Conjecture (3) also reduces the study of generalized L-functions for arbitrary G to the known theory for GL_n.

Though many specific cases of the functoriality conjecture have been verified, it is still far from being solved. For more information, refer to [**1**] and [**9**].

6. Application

Suppose E is a Galois extension of $F = \mathbf{Q}$. We think of E as the splitting field of some monic polynomial $f(x)$ with integer coefficients. Recall that a prime p splits completely in E if and only if $Fr_p = 1$. In terms of the polynomial $f(x)$, it means in general that $f(x)$ splits into linear factors mod p. Let $\mathrm{Spl}(E/\mathbf{Q})$ be the set of primes p which split completely in E. For example we see in §1 that for $E = \mathbf{Q}(\sqrt{-1})$,

$$\mathrm{Spl}(E/\mathbf{Q}) = \{p \mid p \equiv 1 \bmod 4\}.$$

In general, it is known that the map $E \to \mathrm{Spl}(E/\mathbf{Q})$ is an injective order reversing map from finite Galois extensions of $\mathbf{Q}$ into subsets of prime numbers. In other words, the set of splitting primes determines E uniquely. Thus it is natural to ask "What is the image of this map?, i.e., What sets of prime numbers are of the form $\mathrm{Spl}(E/\mathbf{Q})$?" A solution to this problem would constitute some kind of nonabelian class field theory since we would be able to parametrize all the finite Galois extensions E of $\mathbf{Q}$ by the collections $\mathrm{Spl}(E/\mathbf{Q})$.

We close this section by illustrating how Langlands' ideas shed light on this fundamental problem.

Let $\overline{\mathbf{Q}}$ denote an algebraic closure of $\mathbf{Q}$. Given a Galois extension E of $\mathbf{Q}$, there is a homomorphism $\sigma : G(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_n(\mathbf{C})$ with the property that $G(\overline{\mathbf{Q}}/E)$ is the kernel of σ. Hence we get a faithful representation $\sigma : G(E/\mathbf{Q}) \to GL_n(\mathbf{C})$ to which we can attach the Artin L-function $L(s,\sigma)$. Moreover, the definitions are such that $\mathrm{Spl}(E/\mathbf{Q}) = \{p \mid \sigma(Fr_p) = I\}$. Langlands' reciprocity law (Conjecture 1) asserts that the family $\{\sigma(Fr_p)\}$ is automorphic, that is, there exists an automorphic representation $\pi = \otimes\pi_p$ of GL_n over $\mathbf{Q}$ such that for all p outside S_π, $A_p = \sigma(Fr_p)$. In particular, $\mathrm{Spl}(E/\mathbf{Q}) = \{p \mid A_p = I\}$. Therefore the truth of Conjecture 1 reduces the reciprocity problem above to the study of automorphic representations of GL_n. This relation is typical of the perspectives which Langlands' problem brings to classical number theory.

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