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EXTREME POSITIVE OPERATORS ON THE ORDERED SPACE OF 2 × 2 HERMITIAN MATRICES

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1. Introduction

There have been various studies for characterizations of positive linear operators on C^*-algebras. Stormer[**6**] studied those preserving the order identity with additional order properties such as of class 0 or of class 1. Chu and Jefferies [**1**] considered extreme positive linear maps between JB-algebras which preserve the order identity.

Attempting to study the order characteristics of positive linear operators, one evidently comes to study the extreme positive linear operators. In this paper, we consider the finite dimensional case and further restrict ourselves to the ordered space of 2×2 Hermitian matrices while eliminating other order properties imposed on positive linear operators except the extremality.

We prove in Theorem 5.1 that a positive linear operator T is extreme if and only if it is unitarily equivalent to a linear map of the form $S_{\mathbf{z}}$ described below. Thus, T is extreme if and only if T maps every extreme point to either 0 or another extreme point.

Throughout this paper, E will always be used to denote the real ordered space of all 2×2 Hermitian matrices with the positive cone consisting of all elements having nonnegative eigenvalues. An element of E is positive if and only if both of its diagonal entries along with its determinant are nonnegative.

If A is a 2×2 complex matrix, then $\overline{A}$ will be used to denote the complex conjugate of A and A^* for the transpose of $\overline{A}$, i.e. $\overline{A}^T$. We use $\mathbf{e}_i$ for the unit vector in $\mathbb{C}^2$ with 1 in the ith component and zero for the other.

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We denote E_{ii} for $\mathbf{e}_i\mathbf{e}_i^T$, E_{12} for $\mathbf{e}_1\mathbf{e}_2^T+\mathbf{e}_2\mathbf{e}_1^T$ and $\widetilde{E}_{12}$ for $i\mathbf{e}_1\mathbf{e}_2^T-i\mathbf{e}_2\mathbf{e}_1^T$. The unit matrix $E_{11}+E_{22}$ will be denoted by I while I will also be used for the identity operator on E.

Recall that every element of E can be written as $\lambda\mathbf{x}\mathbf{x}^* + \mu\mathbf{y}\mathbf{y}^*$ for some λ, $\mu \in \mathbb{R}$ and $\{\mathbf{x},\mathbf{y}\}$ orthonormal set of eigenvectors. If T is a linear operator on E, then T is determined whenever $T(\mathbf{x}\mathbf{x}^*)$ is defined for every $\mathbf{x} \in \mathbb{C}^2$. We say a linear operator T is positive, i.e. $T \geq 0$ if $T(P) \geq 0$ whenever $P \geq 0$. Note that $T \geq 0$ if and only if $T(\mathbf{x}\mathbf{x}^*) \geq 0$ for all $\mathbf{x} \in \mathbb{C}^2$.

DEFINITION 1.1.. *A linear operator T on E is said to be strongly positive if $T(P) \geq 0$ for all $P \geq 0$ and whenever $T(P) \geq 0$, there exists $Q \geq 0$ such that $T(Q) = T(P)$.*

DEFINITION 1.2.. *A nonzero positive linear operator T is said to be extreme or is said to generate an extreme ray if $S = \lambda T$ for some $\lambda \geq 0$ whenever $0 \leq S \leq T$.*

EXAMPLE 1.3..

(a) *The identity operator I on E is extreme.*
(b) *If $T(A) = \overline{A}$ for all $A \in E$, then T is extreme.*
(c) *If $T(aE_{11} + dE_{22} + bE_{12} + c\widetilde{E}_{12}) = aE_{11} + dE_{22} + bE_{12}$ for all a, b, c, $d \in \mathbb{R}$, then T is not extreme.*

Routine verifications of (a) and (b) are omitted. For (c), we take $S = \frac{1}{2}I$, then $0 \leq S \leq T$ while $S \neq \lambda T$ for any $\lambda \geq 0$.

If Q is an arbitrary nonsingular 2×2 matrix, then we may define a linear operator by $S_Q(A) = QAQ^*$ for all $A \in E$. It is clear that S_Q is one-to-one positive with $S_{Q^{-1}}$ as its inverse. When U is a unitary matrix, we write this operator by U itself instead of S_U for simplicity.

In case $U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ for some $\theta \in \mathbb{R}$, then

$$S_U(A) = UAU^* = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$
$$= \begin{pmatrix} a & (b+ci)e^{-i\theta} \\ (b-ci)e^{i\theta} & d \end{pmatrix}.$$

We write this operator as S_θ instead of S_U or U. Note that $S_U^{-1} = S_{U^{-1}} = S_{U^*}$ for an arbitrary unitary matrix U and $S_\theta^{-1} = S_{-\theta}$.

If $\mathbf{z}$ is an arbitrary vector in $\mathbb{C}^2$, then we define $S_z(\mathbf{xx}^*) = (x_i z_i \bar{x}_j \bar{z}_j)$ where $(x_i z_i \bar{x}_j \bar{z}_j)$ denotes an element of E with (i,j) element $x_i z_i \bar{x}_j \bar{z}_j$. If $\mathbf{z}^T = (p, q)$ where p, $q > 0$, then $S_\mathbf{z}$ is a strongly positive one-to-one linear operator.

THEOREM 1.4. *Let T be a positive linear operator on E with range of T having dimension 1. If T is extreme, then there exist unitary matrices U and V such that $U \circ T \circ V = S_z$ for some $\mathbf{z} \in \mathbb{C}^2$.*

Proof. Note that $T(E)$ is positively generated and hence $T(E) = \{\lambda P \mid \lambda \in \mathbb{R}\}$ for some $P \geq 0$, where P is not positive definite since T is extreme. Let U be a unitary matrix such that $UPU^* = E_{11}$ and let $T_1 = U \circ T$. Then $T_1(A) = \lambda E_{11}$ for every $A \in E$. We define a linear functional on E such that $f(A) = \lambda$ whenever $T_1(A) = \lambda E_{11}$. Clearly, $f \geq 0$ and f is extreme since T_1 is extreme. Therefore, there exists $\mathbf{z} \in \mathbb{C}^2$ such that $f(A) = \mathbf{z}^* A \mathbf{z}$ for all $A \in E$. Let $\mathbf{z}_0 = \mathbf{z}/\|\mathbf{z}\|$ and let $\{\mathbf{z}_0, \mathbf{w}_0\}$ be an orthonormal set. If $V = (\mathbf{z}_0, \mathbf{w}_0)$ and $S = T_1 \circ V = U \circ T \circ V$, then

$$S(\mathbf{xx}^*) = T_1((V\mathbf{x})(V\mathbf{x})^*) = f((V\mathbf{x})(V\mathbf{x})^*)E_{11}$$
$$= \mathbf{z}^*(V\mathbf{x})(V\mathbf{x})^*\mathbf{z}E_{11} = (\mathbf{z}^*V\mathbf{x})(\mathbf{z}^*V\mathbf{x})^*E_{11}$$
$$= \|z\|^2 |x_1|^2 E_{11} = (x_i w_i \bar{x}_j \bar{w}_j), \text{ where } w_1 = \|z\|,\ w_2 = 0.$$

Therefore, $S = S_\mathbf{w}$.

2. Positive Operators with Range of Dimension 2

In this section, we consider positive linear operators on E whose ranges having dimension 2. We prove in Theorem 2.4 that in this case, the operators cannot be extreme.

LEMMA 2.1. *If* $\{\mathbf{x}, \mathbf{y}\}$ *is a linearly independent set in* $\mathbb{C}^2$*, then there exists a nonsingular matrix* Q *such that* $S_Q(\mathbf{x}\mathbf{x}^*) = E_{11}$, $S_Q(\mathbf{y}\mathbf{y}^*) = E_{22}$.

Proof. We take a unitary matrix U such that $U\mathbf{x}\mathbf{x}^*U^* = \|\mathbf{x}\|^2 E_{11}$ and let $U\mathbf{y}\mathbf{y}^*U^* = P$. If $P = \begin{pmatrix} p_1 & p_3 \\ \overline{p}_3 & p_2 \end{pmatrix}$, then $p_2 \neq 0$ since $P \geq 0$ from $\mathbf{y}\mathbf{y}^* \geq 0$ and $\{\mathbf{x}\mathbf{x}^*, \mathbf{y}\mathbf{y}^*\}$ is linearly independent. Note that $p_1 p_2 = |p_3|^2$ since zero is an eigenvalue of $\mathbf{y}\mathbf{y}^*$ and so is of P.

Let $A = \begin{pmatrix} \lambda & q \\ 0 & \mu \end{pmatrix}$, where $\lambda = 1/\|\mathbf{x}\|$, $\mu = -\lambda p_3/p_2$, $\mu = 1/\sqrt{p_2}$. Then by a routine computation, we see that $AE_{11}A^* = \dfrac{1}{\|x\|^2}E_{11}$, and $APA^* = E_{22}$. Now, we define $Q = AU$ to obtain the conclusion.

LEMMA 2.2. *Let* T *be a positive linear operator on* E *with* $\dim(\operatorname{Ker} T) = 2$. *Then there exist unitary matrices* U *and* V *such that for* $S = U \circ T \circ V$*, we have* $(\operatorname{Ker} S)^0 = \operatorname{Span}\{\mathbf{x}\mathbf{x}^*, \mathbf{y}\mathbf{y}^*\}$, $S(E) = \operatorname{Span}\{\mathbf{z}\mathbf{z}^*, \mathbf{w}\mathbf{w}^*\}$ *for some* $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w} \in \mathbb{C}^2$.

Proof. Let $J = \operatorname{Ker} T$. Then J^0 is positively generated since J is a full ideal [3; **Thm 1.7**, Chapter II]. Let $J^0 = \operatorname{Span}\{P_0, Q_0\}$ where P_0, $Q_0 \geq 0$. Replace P_0 by $P_0 + Q_0$ and let $VP_0V^* = D$; diagonal, where V is a unitary matrix. Then $VJ^0V^* = \operatorname{Span}\{D, Q\}$ where $Q = VQ_0V^*$.

Let $\lambda_0 = \max\{\lambda > 0 \mid \lambda Q \leq D\}$. Note that $\lambda_0 \geq 1$ and $D - \lambda_0 Q$ is extreme in E since otherwise $D - \lambda_0 Q$ is positive definite which would imply $\varepsilon Q \leq D - \lambda_0 Q$ for some $\varepsilon > 0$. Therefore, $D - \lambda_0 Q = \mathbf{y}\mathbf{y}^*$ for some $\mathbf{y} \in \mathbb{C}^2$. Similarly, we take $\mu_0 = \max\{\mu > 0 \mid \mu\mathbf{y}\mathbf{y}^* \leq D\}$ and $D - \mu_0\mathbf{y}\mathbf{y}^* = \mathbf{x}\mathbf{x}^*$. Then we have

$$VJ^0V^* = \operatorname{Span}\{\mathbf{x}\mathbf{x}^*, \mathbf{y}\mathbf{y}^*\} = (VJV^*)^0.$$

Now, let $T(E) = \operatorname{Span}\{R_1, R_2\}$ where R_1, $R_2 \geq 0$ and apply a similar argument as above to find a unitary matrix U such that $U(T(E))U^* = \operatorname{Span}\{\mathbf{z}\mathbf{z}^*, \mathbf{w}\mathbf{w}^*\}$. We define $S = U \circ T \circ V^*$. It is routine to verify that $(\operatorname{Ker} S)^0 = VJ^0V^* = \operatorname{Span}\{\mathbf{x}\mathbf{x}^*, \mathbf{y}\mathbf{y}^*\}$ and $S(E) = \operatorname{Span}\{\mathbf{z}\mathbf{z}^*, \mathbf{w}\mathbf{w}^*\}$.

LEMMA 2.3. *Let* S *be a positive linear operator on* E. *If* $S(E) = \operatorname{Span}\{E_{11}, E_{22}\}$ *and* $(\operatorname{Ker} S)^0 = \operatorname{Span}\{E_{11}, E_{22}\}$*, then there exist* α_1,

$\alpha_2,\ \alpha_3,\ \alpha_4 \in \mathbb{R}$ *with* $\alpha_i \geq 0,\ i = 1,2,3,4$ *such that* $S = \sum_{i=1}^{r} \alpha_i T_i$ *where* $T_1(\mathbf{xx}^*) = |x_1|^2 E_{11},\ T_2(\mathbf{xx}^*) = |x_1|^2 E_{22},\ T_3(\mathbf{xx}^*) = |x_2|^2 E_{11},\ T_4(\mathbf{xx}^*) = |x_2|^2 E_{22}$.

Proof. Let $S(E_{11}) = \alpha_1 E_{11} + \alpha_2 E_{22},\ S(E_{22}) = \alpha_3 E_{11} + \alpha_4 E_{22}$. Then we must have $\alpha_i \geq 0,\ i = 1,2,3,4$ since $S \geq 0$. If $\mathbf{x} \in \mathbb{C}^2$ then

$$\begin{aligned} S(\mathbf{xx}^*) &= S(|x_1|^2 E_{11} + |x_2|^2 E_{22}) = |x_1|^2 S(E_{11}) + |x_2|^2 S(E_{22}) \\ &= |x_1|^2(\alpha_1 E_{11} + \alpha_2 E_{22}) + |x_2|^2(\alpha_3 E_{11} + \alpha_4 E_{22}) \\ &= \alpha_1 T(\mathbf{xx}^*) + \alpha_2 T^2(\mathbf{xx}^*) + \alpha_3 T_3(\mathbf{xx}^*) + \alpha_4 T_4(\mathbf{xx}^*). \end{aligned}$$

Therefore, $S = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 + \alpha_4 T_4$.

THEOREM 2.4. *If T is a positive linear operator on E with* $\dim(\operatorname{Ker} T) = 2$, *then T is not extreme.*

Proof. By Lemma 2.2, there exist unitary matrices U, V such that for $T_1 = U \circ T \circ V$, we have $(\operatorname{Ker} T_1)^0 = \operatorname{Span}\{\mathbf{xx}^*, \mathbf{yy}^*\}$ and $T_1(E) = \operatorname{Span}\{\mathbf{zz}^*, \mathbf{ww}^*\}$. Now, by Lemma 2.1, there exist strongly positive one-to-one linear operators S_1 and S_2 such that $S_1(E_{11}) = \mathbf{xx}^*$, $S_1(E_{22}) = \mathbf{yy}^*$, $S_2(\mathbf{zz}^*) = E_{11}$, $S_2(\mathbf{ww}^*) = E_{22}$. We define $S = S_2 \circ T_1 \circ S_1 = S_2 \circ U \circ T \circ V \circ S_1$, then $S(E) = \operatorname{Span}\{E_{11}, E_{22}\}$ and $(\operatorname{Ker} S)^0 = \operatorname{Span}\{E_{11}, E_{22}\}$. Apply Lemma 2.3 to conclude that S is not extreme and hence neither is T.

3. Positive Operators with Range of Dimension 3

In this section, we consider positive linear operators on E whose ranges are of dimension 3. We will prove that any such operator cannot be extreme.

LEMMA 3.1. *Let T be a positive linear operator on E with* $\operatorname{Ker} T \cap K \neq \{0\}$ *where K is the positive cone of E. Then we have* $\dim(\operatorname{Ker} T) \geq 3$.

Proof. Let $0 \neq P \in \operatorname{Ker} T \cap K$. If P is positive definite, then we must have $T = 0$ since $\operatorname{Ker} T$ is an order ideal. Thus, we assume $P = \mathbf{xx}^*$ for some $\mathbf{x} \in \mathbb{C}^2$ with $\mathbf{x}^*\mathbf{x} = 1$. Let U be a unitary matrix so that

$U^*\mathbf{xx}^*U = E_{11}$ and let $S = T \circ U$. Then $S(E_{11}) = 0$. Now, let $S(E_{22}) = Q$, $S(E_{12}) = R_1$, $S(\widetilde{E}_{12}) = R_2$. Then from $S \geq 0$, we have

$$S\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} \geq 0 \quad \text{for all} \quad r \geq 0 \quad \text{and} \quad \theta \in \mathbb{R}.$$

Therefore, $r^2Q + r\cos\theta R_1 + r\sin\theta R_2 \geq 0$ for all $r \geq 0$ and $\theta \in R$, from which we obtain $R_1 = R_2 = 0$, i.e. E_{12}, $\widetilde{E}_{12} \in \operatorname{Ker} S$.

LEMMA 3.2. *Let T be a positive linear operator on E with* $\dim(\operatorname{Ker} T) = 1$. *If $T(\mathbf{xx}^*)$ is positive definite for every $0 \neq \mathbf{x} \in \mathbb{C}^2$ unless $T(\mathbf{xx}^*) = 0$, then T is not extreme.*

Proof. Let $0 \neq A \in \operatorname{Ker} T$ with $A = \lambda_1\mathbf{xx}^* + \lambda_2\mathbf{yy}^*$ where $\{\mathbf{x}, \mathbf{y}\}$ is an orthonormal set of eigenvectors of A. Due to Lemma 3.1, we may assume $\lambda_2 = -1$, $\lambda_1 = \lambda > 0$. Let $U = (\mathbf{x}, \mathbf{y})$ and $T_1 = T \circ U$, then $T_1(E_{22}) = T(\mathbf{yy}^*) = \lambda T(\mathbf{xx}^*) = \lambda T_1(E_{11})$. Now, let $T_1(E_{22}) = P$ then P is positive definite by assumption. Let V be a unitary matrix such that $VPV^* = D$; diagonal with d_1, d_2 as diagonal entries. If Q is the diagonal matrix with $q_1 = 1/\sqrt{d_1}$, $q_2 = 1/\sqrt{d_2}$ then $S_Q(D) = I$.

Let $S = S_Q \circ V \circ T \circ U$. Then $S(E_{11}) = I$, $S(E_{22}) = \lambda I$ and $S(E_{12}) = \begin{pmatrix} a_1 & c \\ c & b_1 \end{pmatrix}$, $S(\widetilde{E}_{12}) = \begin{pmatrix} a_2 & de^{i\tau} \\ de^{-i\tau} & b_2 \end{pmatrix}$, a_i, b_i, c, $d \in \mathbb{R}$. Then

$$T\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \lambda r^2 + r(a_1\cos\theta + a_2\sin\theta) & cr\cos\theta + dr\sin\theta e^{i\tau} \\ cr\cos\theta + dr\sin\theta e^{-i\tau} & 1 + \lambda r^2 + r(b_1\cos\theta + b_2\sin\theta) \end{pmatrix}$$

is positive definite for all $r \geq 0$ and $\theta \in \mathbb{R}$. Now, if

$$f(r,\theta) = \left(\frac{1}{r} + \lambda r + a_1\cos\theta + a_2\sin\theta\right)\left(\frac{1}{r} + \lambda r + b_1\cos\theta + b_2\sin\theta\right)$$

and $g(\theta) = |c\cos\theta + d\sin\theta e^{i\tau}|^2 = \dfrac{c^2+d^2}{2} + \dfrac{c^2-d^2}{2}\cos 2\theta + cd\cos\tau\sin 2\theta$, $h(r,\theta) = f(r,\theta) - g(\theta)$ then $h(r,\theta) > 0$ for all $r \geq 0$, $\theta \in \mathbb{R}$. Let $m = \min\{h(r,\theta) \mid r \geq 0, \theta \in \mathbb{R}\}$ then $m > 0$ since we clearly have

$m = h(r_0, \theta_0)$ for some $r_0 \geq 0$, $\theta_0 \in \mathbb{R}$. Let $\varepsilon > 0$ such that $2\varepsilon k < m$ where $k = \max\{g(\theta) \mid \theta \in \mathbb{R}\}$ and define

$$S_1(E_{11}) = \frac{1}{2}I,\ S_1(E_{22}) = \frac{\lambda}{2}I,\ S_1(E_{12}) = \frac{1}{2}\begin{pmatrix} a_1 & (1+\varepsilon)c \\ (1+\varepsilon)c & b_1 \end{pmatrix},$$

$$S_1(\widetilde{E}_{12}) = \frac{1}{2}\begin{pmatrix} a_2 & (1+\varepsilon)de^{i\tau} \\ (1+\varepsilon)de^{-i\tau} & b_2 \end{pmatrix}.$$

Then it is routine to check that $0 \leq S_1 \leq S$ with $S_1 \neq \lambda S$ for all $\lambda \geq 0$. Therefore, S is not extreme and neither is T.

LEMMA 3.3. *Let T be a positive linear operator on E with $T(E_{11}) = \alpha E_{11}$, $T(E_{22}) = \beta E_{22}$, α, $\beta > 0$. If $\operatorname{Ker} T \neq \{0\}$, then there exist unitary matrices U, V such that $S = U \circ T \circ V$ satisfies $S(E_{11}) = \alpha E_{11}$, $S(E_{22}) = \beta E_{22}$, $S(E_{12}) = \gamma E_{12}$, $S(\widetilde{E}_{12}) = 0$ for some $\gamma \in \mathbb{R}$.*

Proof.
Let $T(E_{12}) = \begin{pmatrix} a_1 & b_1 + c_1 i \\ b_1 - c_1 i & d_1 \end{pmatrix}$, $T(\widetilde{E}_{12}) = \begin{pmatrix} a_2 & b_2 + c_2 i \\ b_2 - c_2 i & d_2 \end{pmatrix}$ where a_i, b_i, c_i, $d_i \in \mathbb{R}$, $i = 1, 2$. Then from $T \geq 0$, we have

$$T\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha + r(a_1\cos\theta + a_2\sin\theta) & r\cos\theta(b_1 + c_1 i) + r\sin\theta(b_2 + c_2 i) \\ r\cos\theta(b_1 - c_1 i) + r\sin\theta(b_2 - c_2 i) & \beta r^2 + r(d_1\cos\theta + d_2\sin\theta) \end{pmatrix}$$

is positive for all $r \geq 0$, $\theta \in \mathbb{R}$. Thus, we have

$$\frac{\alpha}{r} + a_1\cos\theta + a_2\sin\theta \geq 0, \qquad \beta r + d_1\cos\theta + d_2\sin\theta \geq 0$$

for all $r \geq 0$, $\theta \in \mathbb{R}$, from which we obtain $a_1 = a_2 = d_1 = d_2 = 0$. Now, let $b_1 + c_1 i = te^{i\tau}$, $U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\tau} \end{pmatrix}$, $T_1 = U \circ T$. Then $T_1(E_{11}) = \alpha E_{11}$, $T(E_{22}) = \beta E_{22}$, $T_1(E_{12}) = tE_{12}$ and $T_1(\widetilde{E}_{12}) = \begin{pmatrix} 0 & e + fi \\ e - fi & 0 \end{pmatrix}$ for some e, $f \in \mathbb{R}$. For $A \in E$, if

$$A = \begin{pmatrix} a & b + ci \\ b - ci & d \end{pmatrix} \quad \text{then} \quad T_1(A) = \begin{pmatrix} a\alpha & bt + ce + cfi \\ bt + ce - cfi & d\beta \end{pmatrix}.$$

Hence if $A \in \operatorname{Ker} T_1$ then $a = d = 0$, $cf = 0$, $bt + ce = 0$. If $c \neq 0$ then $f = 0$, i.e. $T_1(\widetilde{E}_{12}) = eE_{12}$. If $c = 0$ then $bt = 0$, i.e. $b = 0$ or $t = 0$. If $b = 0$ then $A = c\widetilde{E}_{12}$, i.e. $T_1(\widetilde{E}_{12}) = 0$ and we are done. When $t = 0$, let $e + fi = se^{i\sigma}$ and $U_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\sigma} \end{pmatrix}$, $T_2 = U_1 \circ T_1$ then $T_2(E_{12}) = 0$, $T_2(\widetilde{E}_{12}) = sE_{12}$.

Therefore, in any case, we have an operator S_0 of the form $V \circ T$ where V is a unitary matrix such that $S_0(E_{11}) = \alpha E_{11}$, $S_0(E_{22}) = \beta E_{22}$, $S_0(E_{12}) = tE_{12}$, $S_0(\widetilde{E}_{12}) = sE_{12}$ with t, $s \in \mathbb{R}$.

Now, let $S_0(sE_{12} - t\widetilde{E}_{12}) = 0$ and $s - ti = \rho e^{\lambda i}$, then with $\theta = \lambda - \frac{\pi}{2}$, $S_\theta(sE_{12} - t\widetilde{E}_{12}) = \rho \widetilde{E}_{12}$. Let $U = S_{-\theta} = S_{\frac{\pi}{2} - \lambda}$ and let $S = V \circ T \circ U$ then S satisfies the desired property.

LEMMA 3.4. *Let T be a positive linear operator on E with $T(E_{11}) = \alpha E_{11}$, $T(E_{22}) = \beta E_{22}$, $T(E_{12}) = \gamma E_{12}$, $T(\widetilde{E}_{12}) = 0$ where α, $\beta > 0$, $\gamma \in \mathbb{R}$. Then T is not extreme.*

Proof. From $T \geq 0$, we have for all $r \geq 0$, $\theta \in \mathbb{R}$

$$T \begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{pmatrix} \alpha & \gamma r \cos\theta \\ \gamma r \cos\theta & \beta r^2 \end{pmatrix} \geq 0.$$

Hence, we have $\alpha\beta \geq \gamma^2$. Now, define $S(E_{11}) = \frac{\alpha}{2} E_{11}$, $S(E_{22}) = \frac{\beta}{2} E_{22}$, $S(E_{12}) = \frac{\gamma}{2} E_{12}$, $S(\widetilde{E}_{12}) = \frac{\gamma}{2} \widetilde{E}_{12}$. Then it is routine to verify that $0 \leq S \leq T$ while $S \neq \lambda T$ for any $\lambda \geq 0$.

LEMMA 3.5. *Let T be a linear operator on E with $T(E_{11}) = E_{11}$, $T(\mathbf{x}\mathbf{x}^*) = \mathbf{y}\mathbf{y}^*$ where $\{\mathbf{x}, \mathbf{e}_1\}$, $\{\mathbf{y}, \mathbf{e}_1\}$ are linearly independent sets. If $\dim(\operatorname{Ker} T) = 1$, then T is not extreme.*

Proof. By Lemma 2.1, we find a strongly positive one-to-one linear operator S_R such that $S_R(E_{11}) = E_{11}$, $S_R(\mathbf{x}\mathbf{x}^*) = E_{22}$. Let $Q = R^{-1}$ then $S_Q(E_{11}) = E_{11}$, $S_Q(E_{22}) = \mathbf{x}\mathbf{x}^*$. Similarly, we find S_1 such that $S_1(E_{11}) = E_{11}$, $S_1(\mathbf{y}\mathbf{y}^*) = E_{22}$. Let $S = S_1 \circ T \circ S_Q$, then $S(E_{11}) = E_{11}$, $S(E_{22}) = E_{22}$ and $S \geq 0$. Now, we apply Lemma 3.4 and Lemma 3.3 to conclude that S is not extreme. Therefore T is not extreme.

LEMMA 3.6. *Let T be a positive linear operator on E with $T(E_{11}) = E_{11}$, $T(E_{22}) = P$ where P is positive definite. Then there exists a nonsingular A such that $S = S_A \circ T$ satisfies $S(E_{11}) = E_{11}$, $S(E_{22})$ is diagonal.*

Proof. Let $P = \begin{pmatrix} p_1 & p_3 \\ \bar{p}_3 & p_2 \end{pmatrix}$ and let $A = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$, $q = -\frac{p_3}{p_2}$. Note that $p_2 \neq 0$ since P is positive definite. We find that $AE_{11}A^* = E_{11}$, $APA^* = \begin{pmatrix} d & 0 \\ 0 & p_2 \end{pmatrix}$ where $d = p_1 - q\bar{p}_3$.

LEMMA 3.7. *Let $f(r,\theta) = d_2(1 + d_1 r^2 + kr\sin(\theta+\alpha)) - \ell^2 \sin^2(\theta+\beta)$ where k, d_1, $d_2 > 0$ and let $m = \min\{f(r,\theta) \mid r \geq 0, \theta \in \mathbb{R}\}$. If $m = 0$ with $f(0,\theta_0) = 0$ for some $\theta_0 \in \mathbb{R}$ and $f(r,\theta) \neq 0$ for $r \neq 0$, then $4d_1 - k^2 > 0$ and there exists $\delta > 0$ such that $f(r,\theta) \geq \delta^2 r^2$ for all $r \geq 0$, $\theta \in \mathbb{R}$.*

Proof. It is easy to check that f is bounded below and must assume its minimum at some point. From the assumption, $f(0,\theta) = d_2 - \ell^2 \sin^2(\theta + \beta) \geq 0$ for all $\theta \in \mathbb{R}$. Hence, $d_2 = \ell^2$ and $\theta_0 = -\beta + \frac{\pi}{2} + m\pi$. Also, from $f_r(0,\theta_0) = 0$, we obtain $\sin(\theta_0 + \alpha) = 0$, i.e. $\theta_0 = -\alpha + n\pi$. Therefore, we must have $\beta - \alpha = \frac{\pi}{2} + \ell\pi$ and hence $\sin^2(\theta+\alpha) = \cos^2(\theta+\beta)$ for all $\theta \in \mathbb{R}$. Now,

$$\begin{aligned} f(r,\theta) =& d_2 - \ell^2 + d_1 d_2 r^2 + k d_2 r \sin(\theta+\alpha) + \ell^2 \sin^2(\theta+\alpha) \\ =& d_1 d_2 \left(r^2 + \frac{k}{d_1} r \sin(\theta+\alpha) + \frac{1}{d_1} \sin^2(\theta+\beta) \right) \\ \geq& \frac{d_1 d_2}{4} \left(\frac{4}{d_1} \sin^2(\theta+\alpha) - \frac{k^2}{d_1^2} \sin^2(\theta+\alpha) \right) \\ =& \frac{d_2}{4d_1} (4d_1 - k^2) \sin^2(\theta+\alpha) \end{aligned}$$

where the inequality is taken from the minimum of $f(r,\theta)$ considered as a quadratic function of r, i.e. with $r = -\frac{k}{2d_1}\sin(\theta+\alpha)$. Therefore, we must have $4d_1 - k^2 > 0$ since otherwise $f(r_0,\theta_0) = 0$ for some $r_0 > 0$ and $\theta_0 \in \mathbb{R}$.

Now, choose δ so that $4\delta^2 < (4d_1 - k^2)d_2^2,\ d_1 d_2$. Then

$$\begin{aligned} f(r,\theta) - \delta^2 r^2 &= (d_1 d_2 - \delta^2) r^2 + d_2 k r \sin(\theta+\alpha) + d_2 \sin^2(\theta+\alpha) \\ &\geq \frac{(4d_1 - k^2)d_2^2 - 4\delta^2}{4(d_1 d_2 - \delta^2)} \sin^2(\theta+\alpha) \geq 0 \quad \text{for all} \quad r \geq 0,\ \theta \in \mathbb{R}. \end{aligned}$$

LEMMA 3.8. *Let T be a positive linear operator on E with* $\dim(\operatorname{Ker} T) = 1$, $T(E_{11}) = E_{11}$, $T(E_{22}) = d_1 E_{11} + d_2 E_{22}$, $d_1,\ d_2 > 0$.
If $\dim(\operatorname{Span}\{\mathbf{xx}^* \mid T(\mathbf{xx}^*)$ *is extreme*$\}) = 1$, *then T is not extreme.*

Proof.
Let $T(E_{12}) = \begin{pmatrix} a_1 & b_1 + c_1 i \\ b_1 - c_1 i & f_1 \end{pmatrix}$, $T(\widetilde{E}_{12}) = \begin{pmatrix} a_2 & b_2 + c_2 i \\ b_2 - c_2 i & f_2 \end{pmatrix}$.
Then $f_1 = f_2 = 0$ from $T \geq 0$. By applying a unitary map of the form S_θ, we may assume $c_1 = 0$. Since $\operatorname{Ker} T \neq \{0\}$, we must have $c_2 = 0$. Hence,

$$T\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{pmatrix} 1 + d_1 r^2 + a_1 r\cos\theta + a_2 r \sin\theta & b_1 r\cos\theta + b_2 r\sin\theta \\ b_1 r \cos\theta + b_2 r \sin\theta & d_2 r^2 \end{pmatrix}$$

where the determinant is $r^2 f(r,\theta) = r^2 d_2(1 + d_1 r^2 + kr\sin(\theta+\alpha)) - r^2\ell^2\sin^2(\theta+\beta)$, $k^2 = a_1^2 + a_2^2$, $\ell^2 = b_1^2 + b_2^2$, $\tan\alpha = a_1/a_2$, $\tan\beta = b_1/b_2$.

Let $m = \min\{f(r,\theta) \mid r \geq 0, \theta \in \mathbb{R}\}$. When $m = 0$ with $f(0,\theta_0) = 0$ for some $\theta_0 \in \mathbb{R}$, then by Lemma 3.7, we have $4d_1 - k^2 \gneq 0$. We define $S(E_{11}) = \dfrac{1}{2}E_{11}$, $S(E_{22}) = \dfrac{1}{2}\begin{pmatrix} d_1 & \delta i \\ -\delta i & d_2 \end{pmatrix}$, $S(E_{12}) = \dfrac{1}{2}T(E_{12})$, $S(\widetilde{E}_{12}) = \dfrac{1}{2}T(\widetilde{E}_{12})$. Then

$$2S\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{pmatrix} 1 + d_1 r^2 + rk\sin(\theta+\alpha) & i\delta r^2 + r\ell\sin(\theta+\beta) \\ -i\delta r^2 + r\ell \sin(\theta+\beta) & d_2 r^2 \end{pmatrix}$$

where δ is as defined in Lemma 3.7. Note that the diagonal entries are all nonnegative since $T \geq 0$, and that the determinant is nonnegative due to Lemma 3.7. Therefore, we have $0 \leq S \leq T$ with $S \neq \lambda T$ for any $\lambda \geq 0$.

Next, we consider the case where $m = f(r_0, \theta_0)$ with $r_0 \neq 0$ or with $m > 0$. If $r_0 \neq 0$, then $\dim(\operatorname{Span}\{\mathbf{xx}^* \mid T(\mathbf{xx}^*)$ is extreme$\}) \geq 2$. Thus, we are left with the case where $m > 0$. Then there exists $\varepsilon > 0$ such that $f(r,\theta) \geq f(r_0,\theta_0) > 3\varepsilon(b_1^2+b_2^2)$ for all $r \geq 0$, $\theta \in \mathbb{R}$. Now, we define $S(E_{11}) = \frac{1}{2}E_{11}$, $S(E_{22}) = \frac{1}{2}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, $S(E_{12}) = \frac{1}{2}T(E_{12}) - \frac{b_1\varepsilon}{2}E_{12}$, $S(\widetilde{E}_{12}) = \frac{1}{2}T(\widetilde{E}_{12}) - \frac{b_2\varepsilon}{2}E_{12}$, then $S \geq 0$ since $d_2(1 + d_1r^2 + kr\sin(\theta + \alpha)) - (1-\varepsilon)^2\ell^2\sin^2(\theta+\beta) \geq 0$ for all $r \geq 0$, $\theta \in \mathbb{R}$. Similarly, we can show that $T - S \geq 0$ while $S \neq \lambda T$ for any $\lambda \geq 0$. Therefore, T is not extreme.

THEOREM 3.9. *Let T be a positive linear operator on E. If* $\dim(\operatorname{Ker} T) = 1$, *then T is not extreme.*

Proof. Let $F = \operatorname{Span}\{\mathbf{xx}^* \mid T(\mathbf{xx}^*) \text{ is extreme}\}$. We consider the three cases of $\dim F = 0$, $\dim F = 1$ and $\dim F \geq 2$. When $\dim F = 0$, the theorem follows from Lemma 3.2. We assume next that $\dim F = 1$. Let $T(\mathbf{xx}^*) = \mathbf{zz}^*$ where $\mathbf{x}$ is a unit vector and find unitary matrices U, V such that $U^*\mathbf{xx}^*U = E_{11}$, $V\mathbf{zz}^*V^* = qE_{11}$. Define $X = \frac{1}{q}V \circ T \circ U$ and apply Lemma 3.6 and 3.8 to conclude S is not extreme and hence T is not extreme.

Finally we consider the case where $\dim F \geq 2$. Let $\{\mathbf{xx}^*, \mathbf{yy}^*\}$ be linearly independent such that $T(\mathbf{xx}^*) = \mathbf{zz}^*$, $T(\mathbf{yy}^*) = \mathbf{ww}^*$. In case $\{\mathbf{zz}^*, \mathbf{ww}^*\}$ is linearly dependent, one can show easily that $\dim(\operatorname{Ker} T) \geq 3$. Hence, we may assume $\{\mathbf{zz}^*, \mathbf{ww}^*\}$ is linearly independent. We apply Lemma 2.1 to find one-to-one strongly positive linear operators S_1 and S_2 such that $S_1(E_{11}) = \mathbf{xx}^*$, $S_1(E_{22}) = \mathbf{yy}^*$, $S_2(\mathbf{zz}^*) = E_{11}$, $S_2(\mathbf{ww}^*) = E_{22}$. Let $S = S_2 \circ T \circ S_1$ then by Lemmas 3.3 and 3.4, S is not extreme and hence neither is T.

4. One-to-One Positive Linear Operators

In this section, we will consider one-to-one positive linear operators on E. F will be used to denote the subspace of E spanned by $\{\mathbf{xx}^* \mid T(\mathbf{xx}^*) \text{ is extreme}\}$ where T is the operator being concerned.

LEMMA 4.1. *Let* $g(\theta) = b_1^2\cos^2\theta + b_2^2\sin^2\theta + b_1b_2\cos\tau\sin 2\theta$, $f(r,\theta) = d_2(1 + d_1r^2 + kr\sin(\theta+\alpha)) - g(\theta)$ *where* d_1, d_2, $k > 0$, *and* $q(\theta) =$

$d_2k^2\sin^2(\theta+\alpha)/(d_2-g(\theta))$. *If* $\min\{f(r,\theta)\mid r\geq 0,\theta\in\mathbb{R}\}=f(0,\lambda)=0$ *and if* $f(r_0,\theta)>0$ *for all* $r_0\gneq 0$ *then* $\max\{q(\theta)\mid\theta\in[0,\pi]\}<4d_1$.

Proof. From $f(0,\lambda)=d_2-g(\lambda)$, we have $d_2=g(\lambda)=g(\lambda+n\pi)$. Considering $f(r,\lambda)$ as a quadratic function of r which is nonnegative for all $r\geq 0$, we must have $\sin(\lambda+\alpha)\geq 0$. Similarly, we have $\sin(\lambda+\alpha+\pi)\geq 0$. Therefore, we have $\sin(\lambda+\alpha)=0$, i.e. $\lambda=-\alpha+n\pi$. Note that in $[0,\pi]$, there can be at most one solution for $d_2=g(\theta)$ since $d_2\geq g(\theta)$ for all $\theta\in\mathbb{R}$. Now,

$$\lim_{\theta\to\lambda}q(\theta)=\lim_{\theta\to\lambda}\frac{d_2k^2\sin(2\theta+2\alpha)}{-g'(\theta)}$$

which is 0 if $g'(\lambda)\neq 0$ and is $-2d_2k^2/g''(\lambda)$ if $g'(\lambda)=0$. Note that $g(\theta)$ is a function of the form $A\sin 2\theta+B\cos 2\theta+C$ and hence $g'(\theta),g''(\theta)$ cannot vanish simultaneously.

Let $D(\theta)$ be the discriminant of $f(r,\theta)$ as a quadratic function of r, i.e. $D(\theta)=d_2^2k^2\sin^2(\theta+\alpha)-4d_1d_2(d_2-g(\theta))$. If $D(\mu)>0$ for some μ, then with $r_0=-kd_2\sin(\mu+\alpha)+\sqrt{D(\mu)}$ or with $r_0=-kd_2\sin(\mu+\alpha+\pi)+\sqrt{D(\mu+\pi)}=kd_2\sin(\mu+\alpha)+\sqrt{D(\mu)}$, we have $r_0>0$ and $f(r_0,\mu)=0$. But this is a contradiction to the hypothesis, i.e. we must have $D(\mu)\leq 0$. Similarly, if $D(\mu)=0$ for some μ then from $r_0=0$, we must have $\sin(\mu+\alpha)=0$ and hence $d_2-g(\mu)=0$ from $D(\mu)=0$.

Therefore, for every θ such that $d_2\neq g(\theta)$, we have

$$d_2^2k^2\sin^2(\theta+\alpha)<4d_1d_2(d_2-g(\theta)),\ \text{i.e.}\ q(\theta)<4d_1.$$

Now, we are left to show $q(\lambda)<4d_1$. In case $g'(\lambda)\neq 0$, we have $q(\lambda)=0$ by definition and hence $q(\lambda)<4d_1$. When $g'(\lambda)=0$, note that $D'(\lambda)=d_2^2k^2\sin(2\lambda+2\alpha)+4d_1d_2g'(\lambda)=0$ and hence $D''(\lambda)\neq 0$, i.e. $2d_2^2k^2+4d_1d_2g''(\lambda)\neq 0$ since $D(\theta)$ is a function of the form $A\sin 2\theta+B\cos 2\theta+C$. Therefore,

$$q(\lambda)=\lim_{\theta\to\lambda}\frac{d_2k^2\sin(2\theta+2\alpha)}{-g'(\theta)}=-\frac{2d_2k^2}{g''(\lambda)}\neq 4d_1,$$

i.e. $q(\lambda)<4d_1$. It is clear now that $\max\{q(\theta)\mid\theta\in[0,\pi]\}<4d_1$.

LEMMA 4.2. *Let T be a one-to-one positive linear operator on E with* $\dim F = 1$. *Then T is not extreme.*

Proof. Let $T(\mathbf{xx}^*) = \mathbf{zz}^*$ and find unitary matrices U and V such that $U \circ T \circ V(E_{11}) = \alpha E_{11}$ for some $\alpha > 0$. Let $T_1 = \dfrac{1}{\alpha} U \circ T \circ V$ and apply Lemma 3.6 to find a one-to-one strongly positive S_1 such that $S = S_1 \circ T_1$ satisfies $S(E_{11}) = E_{11}$, $S(E_{22}) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, $d_1, d_2 > 0$. By applying a unitary operator S_θ, we assume that

$$S(E_{12}) = \begin{pmatrix} a_1 & b_1 \\ b_1 & 0 \end{pmatrix}, \quad S(\widetilde{E}_{12}) = \begin{pmatrix} a_2 & b_2 e^{i\tau} \\ b_2 e^{-i\tau} & 0 \end{pmatrix} \quad \text{where} \quad b_1, b_2 \in \mathbb{R}.$$

The zero entries in the above are due to $S \geq 0$. Now,

$$S\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{pmatrix} 1 + d_1 r^2 + a_1 r\cos\theta + a_2 r \sin\theta & b_1 r\cos\theta + b_2 r \sin\theta e^{i\tau} \\ b_1 r \cos\theta + b_2 r\sin\theta e^{-i\tau} & d_2 r^2 \end{pmatrix}$$

whose determinant is $f(r,\theta) = d_2(1 + d_1 r^2 + kr\sin(\theta+\alpha)) - g(\theta)$, where $g(\theta) = b_1^2\cos^2\theta + b_2^2\sin^2\theta + b_1 b_2 \cos\tau \sin 2\theta, k^2 = a_1^2 + a_2^2$. Since $f(r,\theta) \geq 0$ for all $r \geq 0$, $\theta \in \mathbb{R}$, if $m = \min\{f(r,\theta) \mid r \geq 0, \theta \in \mathbb{R}\}$ then $m \geq 0$. Let $M = \max\{g(\theta) \mid \theta \in \mathbb{R}\}$. If $M = 0$, then we have $b_1 = b_2 = 0$ which would imply $\operatorname{Ker} T \neq \{0\}$. Therefore, $M \neq 0$.

First, we assume $m \neq 0$. Choose $\varepsilon > 0$ such that $\varepsilon M < m$ and define

$$S_0(E_{11}) = \frac{1}{2} S(E_{11}), \quad S_0(E_{22}) = \frac{1}{2} S(E_{22}),$$
$$S_0(E_{12}) = \frac{1}{2}\begin{pmatrix} a_1 & (1+\varepsilon)b_1 \\ (1+\varepsilon)b_1 & 0 \end{pmatrix},$$
$$S_0(\widetilde{E}_{12}) = \frac{1}{2}\begin{pmatrix} a_2 & (1+\varepsilon)b_2 e^{i\tau} \\ (1+\varepsilon)b_2 e^{-i\tau} & 0 \end{pmatrix},$$

then by a routine computation, we have $0 \leq S_0 \leq S$ while $S_0 \neq \lambda S$ for any $\lambda \geq 0$, i.e. S is not extreme.

Next, we consider the case of $m = 0$. We must have $m = f(r_0, \lambda)$ for some $r_0 \geq 0$ and $-\pi > \lambda > \pi$. Suppose $r_0 \neq 0$, then $T(\mathbf{yy}^*) = T\begin{pmatrix} 1 & r_0 e^{i\lambda} \\ r_0 e^{-i\lambda} & r_0^2 \end{pmatrix}$ has determinant zero, i.e. $T(\mathbf{yy}^*)$ is extreme, which is contrary to the hypothesis. Therefore, we must have $r_0 = 0$, i.e. $f(0, \lambda) = 0$ for some λ. Now, by Lemma 4.1, if $q(\theta) = d_2 k^2 \sin^2(\theta + \alpha)/(d_2 - g(\theta)), L = \max\{q(\theta) \mid \theta \in [0, \pi]\}$ then $L < 4d_1$. Note also that $1 + d_1 r^2 + kr \sin(\theta + \alpha) \neq 0$ for all $r \geq 0$ and $\theta \in \mathbb{R}$ from which we obtain $d_1 r^2 - kr + 1 > 0$ for all $r \geq 0$ and hence $k^2 < 4d_1$.

Now, choose $\delta > 0$ such that $\delta < \min\{d_2, d_1 - \frac{L}{4}, d_1 - \frac{k^2}{4}\}$ and define $R(E_{11}) = \frac{1}{2}S(E_{11})$, $R(E_{22}) = \frac{1}{2}S(E_{22}) - \frac{\delta}{2}E_{11}$, $R(E_{12}) = \frac{1}{2}S(E_{12})$, $R(\widetilde{E}_{12}) = \frac{1}{2}S(\widetilde{E}_{12})$, then it is routine to verify that $0 \leq R \leq S$ while $R \neq \lambda S$ for any $\lambda \geq 0$. Therefore, S is not extreme and hence T is not extreme.

LEMMA 4.3. *Let T be a positive linear operator on E with $T(E_{ii}) = E_{ii}$, $i = 1, 2$. If $T(E_{12}) = cE_{12}$, $T(\widetilde{E}_{12}) = fE_{12} + g\widetilde{E}_{12}$ where $c, f, g \in \mathbb{R}$, then there exist unitary matrices U, V such that $S = V \circ T \circ U$ satisfies $S(E_{ii}) = E_{ii}$, $i = 1, 2$, $S(E_{12}) = dE_{12}$, $S(\widetilde{E}_{12}) = d\cos\alpha E_{12} + d\sin\alpha\widetilde{E}_{12}$ for some $d > 0$, $\alpha \in \mathbb{R}$.*

Proof. We define τ by $\tan 2\tau = (-c^2 + f^2 + g^2)/2cf$ where $\tau = \pi/4$ when $cf = 0$ and let $S_1 = T \circ U_\tau$. Then we have

$$S_1(E_{12}) = (c\cos\tau + f\sin\tau)E_{12} + g\sin\tau\widetilde{E}_{12}$$
$$S_1(\widetilde{E}_{12}) = (-c\sin\tau + f\cos\tau)E_{12} + g\cos\tau\widetilde{E}_{12}.$$

Note that $(c\cos\tau + f\sin\tau)^2 + (g\sin\tau)^2 = (-c\sin\tau + f\cos\tau)^2 + (g\cos\tau)^2$. Let $c\cos\tau + f\sin\tau + ig\sin\tau = de^{i\sigma}$ and let $S = U_\sigma \circ T \circ U_\tau$. Then $S(E_{12}) = dE_{12}$ and from $(-c\sin\tau + f\cos\tau + ig\cos\tau)e^{-i\sigma} = de^{i\alpha}$ for some $\alpha \in \mathbb{R}$, we have $S(\widetilde{E}_{12}) = d\cos\alpha E_{12} + d\sin\alpha\widetilde{E}_{12}$.

LEMMA 4.4. *Let T be a one-to-one positive linear operator on E with $T(E_{ii}) = E_{ii}$, $i = 1, 2$, $T(E_{12}) = cE_{12}$, $T(\widetilde{E}_{12}) = c\cos\tau E_{12} + c\sin\tau\widetilde{E}_{12}$ where $c > 0$, $\pi \geq \tau \geq -\pi$. Then T is extreme if and only if $c = 1$ and $\tau = \frac{\pi}{2}$ or $-\frac{\pi}{2}$.*

Proof. If part is trivial as noted in Examples 1.3. For the only if part, note that we have $c^2(1 + \sin 2\theta \cos\tau) \leq 1$ for all $\theta \in \mathbb{R}$ from $T \geq 0$.

Thus, we have $c^2(1+|\cos\tau|) \le 1$. In case $c^2(1+|\cos\tau|) < 1$, we can find $\varepsilon > 0$ such that $(1+\varepsilon)^2c^2(1+|\cos\tau|) < 1$ and define $S(E_{ii}) = \frac{1}{2}T(E_{ii})$, $i = 1,\ 2$, $S(E_{12}) = \frac{1-\varepsilon}{2}T(E_{12})$, $S(\widetilde{E}_{12}) = \frac{1-\varepsilon}{2}T(\widetilde{E}_{12})$. Then we have $0 \le S \le T$ with $S \ne \lambda T$ for any $\lambda \ge 0$, i.e. T is not extreme.

Thus, we assume $c^2(1+|\cos\tau|) = 1$. Since T is extreme if and only if $\overline{T}$ is extreme, we may further assume $0 \le \tau \le \pi$. First, consider the case $0 \le \tau \le \frac{\pi}{2}$ so that $\cos\tau \ge 0$ and let $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{-\frac{\pi}{4}i} & -e^{-\frac{\pi}{4}i} \end{pmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{\frac{\tau}{2}i} \\ 1 & -e^{\frac{\tau}{2}i} \end{pmatrix}$. If $S = V \circ T \circ U$, then by a routine computation, we have $S(E_{ii}) = E_{ii}$, $i = 1,\ 2$, $S(E_{12}) = E_{12}$, $S(\widetilde{E}_{12}) = \tan\frac{\tau}{2}\widetilde{E}_{12}$. Note that $\tan\tau/2 \le 1$. Now it is easy to see that $\tau = \frac{\pi}{2}$ in order to have S to be extreme.

Next, we consider the case where $\frac{\pi}{2} \le \tau \le \pi$. We repeat the same process with $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{\frac{\pi}{4}i} & -e^{\frac{\pi}{4}i} \end{pmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -ie^{i\tau/2} \\ 1 & ie^{i\tau/2} \end{pmatrix}$ to find $S(\widetilde{E}_{12}) = \cot\frac{\tau}{2}\widetilde{E}_{12}$. Again, we must have $\cot\frac{\tau}{2} = 1$ in order to have S extreme. Therefore $\tau = \frac{\pi}{2}$ in any case and we obtain $c = 1$ from $c^2(1+\cos\tau) = 1$.

LEMMA 4.5. *Let T be a one-to-one positive linear operator on E with $\dim F = 0$. Then T is not extreme.*

Proof. From $\dim F = 0, T(\mathbf{x}\mathbf{x}^*)$ is positive definite for all $0 \ne \mathbf{x} \in \mathbb{C}^2$. Let $T(E_{11}) = P$ and U be a unitary matrix such that $UPU^* = p_1E_{11} + p_2E_{22}$. Let $\mathbf{z}^T = (1/\sqrt{p_1}, 1/\sqrt{p_2})$ and $S_1 = S_{\mathbf{z}} \circ U \circ T$, then $S_1(E_{11}) = I$; the identity matrix. Now, let $S_1(E_{22}) = Q$, $VQV^* = q_1E_{11} + q_2E_{22}$ where V is a unitary matrix and let $S_2 = V \circ S_1$. Then $S_2(E_{11}) = I$, $S_2(E_{22}) = q_1E_{11} + q_2E_{22}$. If

$$S_2(E_{12}) = \begin{pmatrix} a_1 & b_1e^{i\alpha} \\ b_1e^{-i\alpha} & d_1 \end{pmatrix}, \qquad S_2(\widetilde{E}_{12}) = \begin{pmatrix} a_2 & b_2e^{i\beta} \\ b_2e^{-i\beta} & d_2 \end{pmatrix}$$

and if $S_3 = S_\alpha \circ S_2$ then $S_3(E_{11}) = I$, $S_3(E_{22}) = S_2(E_{22})$,

$$S_3(E_{12}) = \begin{pmatrix} a_1 & b_1 \\ b_1 & d_1 \end{pmatrix}, \qquad S_3(\widetilde{E}_{12}) = \begin{pmatrix} a_2 & b_2e^{i\gamma} \\ b_2e^{-i\gamma} & d_2 \end{pmatrix}.$$

Now,

$$S_3\begin{pmatrix} 1 & re^{i\theta} \\ re^{-i\theta} & r^2 \end{pmatrix} = \begin{pmatrix} 1+q_1r^2+kr\sin(\theta+\alpha) & b_1r\cos\theta+b_2r\sin\theta e^{i\gamma} \\ b_1r\cos\theta+b_2r\sin\theta e^{-i\gamma} & 1+q_2r^2+\ell r\sin(\theta+\beta) \end{pmatrix}$$

where $k^2 = a_1^2 + a_2^2$, $\ell^2 = d_1^2 + d_2^2$, $\tan\alpha = a_1/a_2$, $\tan\beta = d_1/d_2$. Let $f(r,\theta) = (1+q_1r^2+kr\sin(\theta+\alpha))(1+q_2r^2+\ell r\sin(\theta+\beta)) - g(\theta)$, where $g(\theta) = |b_1\cos\theta + b_2\sin\theta e^{i\gamma}|^2$. If $m = \min\{f(r,\theta) \mid r \geq 0, \theta \in \mathbb{R}\}$, then $m \neq 0$ since otherwise $\dim F \geq 1$. Let $L = \max\{g(\theta) \mid \theta \in R\}$ and choose $\varepsilon > 0$ such that $\varepsilon L < m$ and define $T_1(E_{ii}) = \frac{1}{2}S_3(E_{ii})$, $T_1(E_{12}) = \frac{1}{2}S_3(E_{12}) + \frac{\varepsilon b_1}{2}E_{12}$, $T_1(\widetilde{E}_{12}) = \frac{1}{2}\begin{pmatrix} a_2 & (1+\varepsilon)b_2e^{i\gamma} \\ (1+\varepsilon)b_2e^{-i\gamma} & d_2 \end{pmatrix}$. Then $0 \leq T_1 \leq S_3$ with $T_1 \neq \lambda S_3$ for any $\lambda \geq 0$. Therefore, S_3 is not extreme and neither is T.

LEMMA 4.6. *Let A be a nonsingular 2×2 matrix. Then there exist unitary matrices U and V such that $S = V \circ S_A \circ U$ satisfies $S(E_{ii}) = d_iE_{ii}$, $i = 1, 2$, $S(E_{12}) = cE_{12}$, $S(\widetilde{E}_{12}) = t\cos\tau E_{12} + t\sin\tau \widetilde{E}_{12}$ for some $d_i > 0$, c, t, $\tau \in \mathbb{R}$.*

Proof. Let $\{\mathbf{x},\mathbf{y}\}$ be an orthonormal set of eigenvectors of A^*A, then $(A\mathbf{x})^*(A\mathbf{y}) = \mathbf{x}^*(A^*A)\mathbf{y} = 0$, i.e. $\{A\mathbf{x}, A\mathbf{y}\}$ is orthogonal. Now, let $U = (\mathbf{x},\mathbf{y})$, $V_1 = (\mathbf{z},\mathbf{w})^*$, $S_1 = V_1 \circ S_A \circ U$ where $\mathbf{z} = A\mathbf{x}/\|A\mathbf{x}\|$, $\mathbf{w} = A\mathbf{y}/\|A\mathbf{y}\|$. Then we have

$$\begin{aligned} S_1(E_{11}) &= V_1 \circ S_A((U\mathbf{e}_1)(U\mathbf{e}_1)^*) = V_1 \circ S_A(\mathbf{x}\mathbf{x}^*) = V_1((A\mathbf{x})(A\mathbf{x})^*) \\ &= \|A\mathbf{x}\|^2 V_1(\mathbf{z}\mathbf{z}^*) = \|A\mathbf{x}\|^2(V_1\mathbf{z})(V_1\mathbf{z})^* = \|A\mathbf{x}\|^2 E_{11} \end{aligned}$$

and similarly, $S_1(E_{22}) = \|A\mathbf{y}\|^2E_{22}$. From $S_1 \geq 0$, we have

$$S_1(E_{12}) = \begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}, \qquad S_1(\widetilde{E}_{12}) = \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}$$

for some α, $\beta \in \mathbb{C}$. Finally, if $\alpha = ce^{it}$ and if $S = S_t \circ S_1$ then S satisfies the desired property.

THEOREM 4.7. *Let T be a one-to-one positive linear operator on E. If T is extreme, then there exist unitary matrices U, V and $\mathbf{z} \in \mathbb{C}^2$ such that $T = U \circ S_z \circ V$ or $\overline{T} = U \circ S_z \circ V$.*

Proof. Let $F = \text{Span}\,\{\mathbf{xx}^* \mid T(\mathbf{xx}^*) \text{ is extreme}\}$, then from 4.2 and 4.5, we must have $\dim F \geq 2$. Let $T(\mathbf{xx}^*) = \mathbf{zz}^*, T(\mathbf{yy}^*) = \mathbf{ww}^*$ where $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent and apply Lemma 2.1 to find one-to-one strongly positive linear operators S_A, S_B such that $T_1 = S_A \circ T \circ S_B$ satisfies $T_1(E_{ii}) = E_{ii}$, $i = 1,\ 2$. From $T_1 \geq 0$, we have $T_1(E_{12}) = aE_{12} + b\widetilde{E}_{12}$, $T_1(\widetilde{E}_{12}) = cE_{12} + d\widetilde{E}_{12}$ for some a, b, c, $d \in \mathbb{R}$.

Let $a + bi = te^{i\tau}$ and $T_2 = S_\tau \circ T_1$, then $T_2(E_{12}) = tE_{12}$ and $T_2(\widetilde{E}_{12}) = fE_{12} + g\widetilde{E}_{12}$. By Lemma 4.3, we find unitary matrices U_0, V_0 such that for $T_3 = U_0 \circ T_2 \circ V_0$, we have $T_3(E_{ii}) = E_{ii}$, $i = 1,\ 2$, $T_3(E_{12}) = sE_{12}$, $T_3(\widetilde{E}_{12}) = s\cos\tau E_{12} + s\sin\tau\widetilde{E}_{12}$. Now, we apply Lemma 4.4 for T_3 so that we have $T_3 = I$ or $\overline{I}$.

Consider the case of $T_3 = I$. Then $U_0 \circ S_\tau \circ S_A \circ T \circ S_B \circ V_0 = I$ from which we obtain $T = S_{A^{-1}} \circ S_{-\tau} \circ U_0^* \circ V_0^* \circ S_{B^{-1}} = S_c$ where $c = A^{-1}WB^{-1}$, $W = U_{-\tau} \circ U_0^* \circ V_0^*$. We apply Lemma 4.6 to find unitary matrices U_1, V_1 such that $S_1 = U_1 \circ T \circ V_1 = U_1 \circ S_c \circ V_1$ satisfies $S_1(E_{ii}) = d_i E_{ii}$, $i = 1,\ 2$, $S_1(E_{12}) = sE_{12}$, $S_1(\widetilde{E}_{12}) = t\cos\tau E_{12} + t\sin\tau\widetilde{E}_{12}$ where s, t, $\tau \in \mathbb{R}$. Let $\mathbf{z}^T = (1/\sqrt{d_1}, 1/\sqrt{d_2})$ and let $S = S_z \circ S_1$. We apply Lemma 4.4 and Lemma 4.3 again for S to obtain $U_2 \circ S_z \circ U_1 \circ T \circ V_1 \circ V_2 = I$. Therefore, we finally have $T = S_c = U_1^* \circ S_w \circ U_2^* \circ V_2^* \circ V_1^* = U \circ S_\mathbf{w} \circ V$ where $\mathbf{w}^T = (\sqrt{d_1}, \sqrt{d_2})$ and $U = U_1^*$, $V = U_2^* \circ V_2^* \circ V_1^*$.

In case $T_3 = \overline{I}$, we use the fact that $\overline{S \circ T} = \overline{S} \circ T$ for any linear operators S and T. We replace U_0 by $\overline{U}_0$ in T_3 and repeat the same process with $\overline{T}_3 = I$.

5. Results and Examples

THEOREM 5.1. *Let T be an arbitrary positive linear operator on E. Then T is extreme if and only if there exist unitary matrices U, V and $\mathbf{z} \in \mathbb{C}^2$ such that $T = U \circ S_z \circ V$ or $\overline{T} = U \circ S_z \circ V$.*

Proof. Only if part is proved by Theorem 1.4, 2.4, 3.9 and 4.7. For the if part, it is sufficient to prove that $S_{\mathbf{z}}$ is extreme for an arbitrary $\mathbf{z} \in \mathbb{C}^2$. First, we consider the case where both components z_1, z_2 of $\mathbf{z}$ are nonzero. Let $0 \leq T \leq S_{\mathbf{z}}$ then $T(E_{11}) = \alpha|z_1|^2 E_{11}$, $T(E_{22}) = \beta|z_2|^2 E_{22}$,

$$T\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \gamma_1 \mathbf{z}\mathbf{z}^*, \quad T\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \gamma_2 \begin{pmatrix} |z_1|^2 & -z_1\bar{z}_2 \\ -\bar{z}_1 z_2 & |z_2|^2 \end{pmatrix},$$

$$T\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \delta \begin{pmatrix} |z_1|^2 & i z_1\bar{z}_2 \\ -i\bar{z}_1 z_2 & |z_2|^2 \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} 2T(E_{12}) &= T((\mathbf{e}_1 + \mathbf{e}_2)(\mathbf{e}_1 + \mathbf{e}_2)^T) - T((\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2)^T)) \\ &= \begin{pmatrix} (\gamma_1 - \gamma_2)|z_1|^2 & (\gamma_1 + \gamma_2) z_1 \bar{z}_2 \\ (\gamma_1 + \gamma_2)\bar{z}_1 z_2 & (\gamma_1 - \gamma_2)|z_2|^2 \end{pmatrix}. \end{aligned}$$

Substituting this into $T((\mathbf{e}_1 + \mathbf{e}_2)(\mathbf{e}_1 + \mathbf{e}_2)^T) = \alpha|z_1|^2 E_{11} + \beta|z_2|^2 E_{22} + T(E_{12}) = \gamma_1 \mathbf{z}\mathbf{z}^*$, we obtain $\alpha = \beta = \gamma_1 = \gamma_2$. Similarly, we can show $\alpha = \beta = \delta$. Therefore, we have $\alpha = \beta = \gamma_1 = \gamma_2 = \delta$, i.e. $T = \alpha S_{\mathbf{z}}$.

COROLLARY 5.2. *Let T be a nonzero positive linear operator on E. Then T is extreme if and only if T maps every extreme point of E to either 0 or another extreme point.*

Proof. Only if part is clear from Theorem 5.1. For the if part, we first consider the case where $T(\mathbf{y}\mathbf{y}^*) = 0$ for some $\mathbf{y} \neq 0$. Find $\mathbf{x}$ such that $\{\mathbf{x}, \mathbf{y}\}$ is orthonormal then $T(\mathbf{x}\mathbf{x}^*) = \mathbf{z}\mathbf{z}^*$ for some $\mathbf{z} \neq 0$ since $T \neq 0$. Now, for some unitary matrices U and V, $S = qU \circ T \circ V$ satisfies $S(E_{11}) = E_{11}, S(E_{22}) = 0$. Note that $\operatorname{Ker} S$ is a full ideal containing E_{22} and hence E_{12}, $\widetilde{E}_{12} \in \operatorname{Ker} S$. Therefore, we have $S = S_z$ with $\mathbf{z}^T = (1, 0)$.

Next, we consider the case where $T(\mathbf{x}\mathbf{x}^*)$ is nonzero for every $0 \neq \mathbf{x} \in \mathbb{C}^2$. It is easy to check that T is one-to-one in this case. By Lemma 2.1, we fine S_A, S_B such that $S_1 = S_A \circ T \circ S_B$ satisfies $S_1(E_{ii}) = E_{ii}$, $i = 1, 2$. By Lemma 4.3, we find U, V such that $S = U \circ S_1 \circ V$ satisfies $S(E_{ii}) = E_{ii}$, $i = 1, 2$, $S(E_{12}) = cE_{12}$, $S(\widetilde{E}_{12}) = c\cos\alpha E_{12} + c\sin\alpha \widetilde{E}_{12}$. Note that both S_1 and S maps every extreme point of E to another extreme point of E and hence $S = I$ or $\overline{I}$ by Lemma 4.4. Therefore S is extreme and so is T.

The following examples of positive linear operators are not extreme. One can show these by direct calculations but in the following, we use theorems proved earlier.

EXAMPLE 5.3. *Let* $T\begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & d \end{pmatrix}$ *for all* a, b, c, $d \in \mathbb{R}$. *Then* T *is not extreme since* $\dim(\operatorname{Ker} T) = 1$.

EXAMPLE 5.4. *Let* $T(E_{11}) = I$, $T(E_{22}) = \frac{1}{2}I$, $T(E_{12}) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $T(\widetilde{E}_{12}) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. *Suppose* T *were extreme and let* $T = U \circ S_{\mathbf{z}} \circ V$ *where* U, V *are unitary and* $\mathbf{z} \in \mathbb{C}^2$. *Then* $T \circ V^* = U \circ S_{\mathbf{z}}$. *If* $\mathbf{x} = V\mathbf{e}_1$ *then* $T \circ V^*(\mathbf{x}\mathbf{x}^*) = T((V^*\mathbf{x})(V^*\mathbf{x})^*) = T(E_{11}) = I$, *while* $U \circ S_z(\mathbf{x}\mathbf{x}^*) = U(\mathbf{y}\mathbf{y}^*) = \mathbf{w}\mathbf{w}^*$ *for some* $\mathbf{w} \in \mathbb{C}^2$. *But* $I \neq \mathbf{w}\mathbf{w}^*$ *for any* $\mathbf{w} \in \mathbb{C}^2$. *Therefore* T *is not extreme.*

EXAMPLE 5.5. *Let* $T\begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix} = \begin{pmatrix} a & c \\ c & d \end{pmatrix}$ *for all* a, b, c, $d \in \mathbb{R}$. T *is not extreme since* $\dim(\operatorname{Ker} T) = 1$. *But note also that* $T\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *i.e.* T *maps extreme point to a non-extreme point.*

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