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GROUP ACTIONS ON HOMOTOPY COMPLEX PROJECTIVE SPACES

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1. Introduction

The purpose of this article is to survey recent developments of group actions on homotopy complex projective spaces and to raise some possible questions one might ask for further research on the subject.

In transformation group theory we usually proceed as follows. We choose a familiar space Y with a well understood group action of G as a model space. Suppose X is a space close to Y, for instance, X is homotopy or cohomology equivalent to Y. We then study when X admits an effective action of G, and in this case how different is the G action on X from that of the model space.

In this paper we set the model space to be a complex projective space $P(\mathbf{C}^n)$ with a linear action of G, and try to understand smooth G actions on manifolds X which are homotopy or cohomology equivalent to $P(\mathbf{C}^n)$.

The following result of Bredon and Su gives an important necessary condition on existence of $\mathbf{Z}_p$ action on $\bmod p$ cohomology $P(\mathbf{C}^n)$ where p is an odd prime.

THEOREM 2.3. [Br]. *Let X be a smooth $\mathbf{Z}_p$ manifold with* $\bmod p$ *cohomology $P(\mathbf{C}^n)$. Let $F_1, \ldots, F_l$ be the fixed point components. Then each F_i is a smooth manifold with* $\bmod p$ *cohomology $P(\mathbf{C}^{n_i})$, and the inclusion $j_i : F_i \longrightarrow X$ induces a surjection* $\mathrm{H}^*(X\,;\mathbf{Z}_p) \longrightarrow \mathrm{H}^*(F_i\,;\mathbf{Z}_p)$ *in the* $\bmod p$ *cohomology. In addition $\sum_{i=1}^{l} n_i = n$.*

In loose terms, Theorem 2.3 says that any $\mathbf{Z}_p$ actions on $\bmod p$ cohomology $P(\mathbf{C}^n)$ resemble linear actions as far as $\bmod p$ cohomology groups are concerned.

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It is known that there are infinitely many homotopy $P(\mathbf{C}^n)$, and they are distinguished up to finite ambiguity by their Pontrjagin classes for $n \geq 4$, [Wa]. In other words there are infinitely many smooth manifolds which are not diffeomorphic to each other but homotopy equivalent to $P(\mathbf{C}^n)$ for $n \geq 4$. Moreover there are only finitely many homotopy $P(\mathbf{C}^n)$'s with the same Pontrjagin classes.

Here we are interested in the following question: For a given group G which homotopy $P(\mathbf{C}^n)$ admits an effective action of G? In this case how different is the action from a linear action?

One of long standing conjectures on group actions on $P(\mathbf{C}^n)$ is the following Petrie's conjecture [Pe]:

Petrie's Conjecture. *If a homotopy $P(\mathbf{C}^n)$ X admits an effective action of S^1, then the total Pontrjagin class $p(X)$ of X is the same as that of $P(\mathbf{C}^n)$, which is $(1+x^2)^n \in \mathrm{H}^*(X\,;\,\mathbf{Z})$. Here $x \in \mathrm{H}^2(X\,;\,\mathbf{Z})$ is a generator of $\mathrm{H}^2(X\,;\,\mathbf{Z}) \cong \mathbf{Z}$.*

If this conjecture is true, only finitely many homotopy $P(\mathbf{C}^n)$ can admit effective S^1 actions. The conjecture is trivially true for $n = 1, 2, 3$, and proved to be true for $n = 4$ [De] and $n = 5$ [Ja]. But it is not known in general whether the conjecture is true or not.

Finite cyclic group version of Petrie's conjecture is still interesting. Namely, if a homotopy $P(\mathbf{C}^n)$ X admits a finite cyclic group action, then is $p(X) = (1+x^2)^n \in \mathrm{H}^*(X\,;\,\mathbf{Z})$? Theorem 4.2 and Theorem 4.3 show that there are infinitely many homotopy $P(\mathbf{C}^n)$ where finite cyclic groups can act effectively with isolated fixed points. This shows that the finite cyclic group version of Petrie's conjecture is not true for actions with isolated fixed points.

Group actions on homotopy $P(\mathbf{C}^n)$ with isolated fixed points are of one extreme type. The other extreme type is the case when the fixed point set consists of a condimension 2 submanifold F and an isolated point P. Such action is called a type II_0 action. Theorem 5.4, Theorem 5.6, and Theorem 5.7 are results about type II_0 actions. Theorem 5.4 shows that if cyclic groups of prime order act on low dimensional homotopy $P(\mathbf{C}^n)$ X as type II_0 actions, then the actions are algebraically standard. The precise definition of algebraically standard action is given in Section 4. In particular, algebraically standardness implies that $p(X) = (1+x^2)^n$, which is the total Pontrjagin class of the standard $P(\mathbf{C}^n)$.

Theorem 5.7 shows that there is an integer $c(X)$ which depends on $p(X)$ such that any type II_0 actions of $\mathbf{Z}_p$ on homotopy $P(\mathbf{C}^n)\, X$ is algebraically standard if $p \geq c(X)$. If S^1 acts on a homotopy $P(\mathbf{C}^n)$ X with type II_0, then any cyclic subgroup $\mathbf{Z}_p \subset S^1$ also acts on X with type II_0. If p is large, then the action is algebraically standard. In particular, the total Pontrjagin class $p(X) = (1+x^2)^n \in \mathrm{H}^*(X\,;\mathbf{Z})$, where $x \in \mathrm{H}^2(X\,;\mathbf{Z})$ is a generator. This shows that Petrie's conjecture for type II_0 action of S^1 is true. This is a known fact, but here we provide a new proof for that.

In contrast to Theorem 5.7, Theorem 5.8 shows a different aspect of type II_0 action. Namely if p is relatively small to compare with the dimension of X, then there are infinitely many X with type II_0 actions of $\mathbf{Z}_p$ which are not algebraically standard.

In this paper we study two extreme types of group actions on homotopy complex projective spaces, i.e., one is an action with isolated fixed points and the other is a type II_0 action.

There are many different types of actions in between, and not much is known for them. One might attempt to find relations between codimension of fixed point sets and algebraically standardness of group actions. In other word, since the phenomina of algebraically standardness appears when the codimension of fixed point set is minimal, and it breaks down when the codimension of fixed point set is maximal, one might attempt to see what happens in between.

This paper is organized as follows. In Section **2** we discuss some basic facts about linear actions on $P(\mathbf{C}^n)$. In Section **3** we discuss some background materials such as G surgery theory, and Atiyah-Singer G signature formula. These are basic tools for later developments. In Section **4** we study cyclic group actions with isolated fixed points. In Section **5** we study type II_0 actions of cyclic groups.

2. Linear Actions on Complex Projective Spaces

Let V be an n-dimensional vector space over the field $\mathbf{C}$ of complex numbers and let $\mathrm{GL}(V)$ be the group of isomorphisms of V onto itself. Then the group $\mathrm{GL}(V)$ is identified with $\mathrm{GL}(n, \mathbf{C})$, the group of invertible square matrices of order n. Let G be a group. Suppose we are given a homomorphism $\rho : \mathrm{G} \longrightarrow \mathrm{GL}(V)$. Then V together with the homomorphism ρ is called a complex representation of G. If V is a real vector space, then $\rho : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ is called a real representation of G. If W is a subspace of V and invariant under the action of G, i.e., $x \in W$ implies $\rho(g)(x) \in W$ for all $g \in G$, then W with the restriction $\rho^W : \mathrm{G} \longrightarrow \mathrm{GL}(W)$ is called a subrepresentation of V. If 0 is the only subrepresentation of V, then V is called irreducible.

EXAMPLE 2.1: Let G be the cyclic group of order m. A complex 1 dimensional irreducible representation t^s is defined as follows. The underlying vector space is $\mathbf{C}$. Embed G in S^1 so that $G \cong \{\exp \frac{2\pi ki}{m} \mid k = 1, \ldots, m\}$. Define $\rho : \mathrm{G} \longrightarrow \mathrm{GL}(\mathbf{C})$ as follows. Let $x \in \mathbf{C}$. Then $\rho(\exp \frac{2\pi ki}{m})(x) = \exp \frac{2\pi ksi}{m} \cdot x$. In other words, a generator $\exp \frac{2\pi i}{m}$ of G acts on $\mathbf{C}$ as the rotation by $\frac{2\pi s}{m}$.

Two representation V and W are equivalent if there exists an isomorphism $f : \mathrm{GL}(V) \longrightarrow \mathrm{GL}(W)$ which makes the following diagram commutative.

$$\begin{array}{ccc} G & \xrightarrow{\rho_V} & \mathrm{GL}(V) \\ & \searrow^{\rho_W} & \downarrow f \\ & & \mathrm{GL}(W) \end{array}$$

Note that if V and W are representations of G, then $V \oplus W$ and $V \otimes W$ are also representations of G under the following induced homomorphisms : if we have $\rho_V : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ and $\rho_W : \mathrm{G} \longrightarrow \mathrm{GL}(W)$, then $\rho_{V\oplus W} : G \rightarrow \mathrm{GL}(V \oplus W)$ and $\rho_{V\otimes W} : G \rightarrow \mathrm{GL}(V \otimes W)$ are defined by $\rho_{V\oplus W}(g)(v, w) = \big(\rho_V(g)(v), \rho_W(g)(w)\big)$ and $\rho_{V\otimes W}(g)(v \otimes w) = \rho_V(g)(v) \otimes \rho_W(g)(w)$ for $g \in G$, $v \in V$, and $w \in W$.

Let $R(G)$ be the complex representation ring of G which is defined as follows. $\mathrm{R}(G) = \{V - W \mid V \text{ and } W \text{are representations of } G\}/\sim$ where $(V - W) \sim (V' - W')$ if there exist some representations S and T such that $V \oplus S$ and $V' \oplus T$ are equivalent and $W \oplus S$ and $W' \oplus T$ are equivalent.

It is not difficult to see that $\mathrm{R}(G)$ forms a ring under the direct sum $\oplus$ and the tensor product $\otimes$.

If $G = \mathbf{Z}_m$ is a cyclic group of order m, then t^s are the only complex irreducible representations for $s = 1, \dots, m$ and

$$\mathrm{R}(G) \cong \mathbf{Z}[t]/(t^m - 1), \quad \text{see [Se]}.$$

Thus any complex representation of G can be expressed as $\sum_{s=1}^{m} a_s t^s$ for $a_s \in \mathbf{Z}$.

We use the notation $P(\mathbf{C}^n) = S(C^n)/S^1$. for the complex projective spaceof dimension $2n-2$. In section **5** we also use the notation $\mathbf{C}P^{n-1}$ for $P(\mathbf{C}^n)$.

For a representation V of G, $S(V)$ is the unit sphere of V. If G is a compact Lie group, then V is equivalent to an orthogonal representation $\rho : G \longrightarrow O(V)$. Thus $S(V)$ is invariant under the action of G. Then $P(V) = S(V)/S^1$ is a complex projective space with a G action. Such an action is called a linear actions on complex projective space.

Let X be a G-manifold. Let G_x be the isotropy subgroup of $x \in X$ of G. Then the tangent space T_xX has a group representation structure of G_x. Indeed, $\rho : G_x \longrightarrow \mathrm{GL}(T_xX)$ is defined by $\rho(g)(v) = Dg\mid_x (v)$. Here Dg is the differential of g. We call it the isotropy representation at x.

PROPOSITION 2.2. *Let G be a cyclic group of odd order m. Let $V = \sum_{s=1}^{m} a_s t^s$ be a complex representation of G. Then*

a) $P(V)^G = \coprod_{s=1}^{m} P(a_s t^s) = \coprod_{s=1}^{m} P(\mathbf{C}^{a_s})$, *where $\coprod$ denotes disjoint union.*

b) *Let $p \in P(a_k t^k) \subset P(V)^G$. Then the isotropy representation at p is $T_pP(V) = \sum_{s \neq k} a_s t^{s-k}$.*

Proof. a) follows from looking at the action in homogeneous coordinates.

b) There exists i such that the point $p_i = [0 : 0 : \cdots 1 : 0 \cdots 0]$ (1 in i-th coordinate) lies in $P(a_k t^k)$. It is an elementary fact that the isotropy representations

$$T_pP(V) = T_{p_i}P(V)$$

By using the coordinate chart around p_i,

$$[z_0 : \cdots : z_n] \longrightarrow (z_0/z_i, \cdots, \widehat{z_i/z_i}, \cdots z_n/z_i) \in \mathbf{C}^n$$

we can see that $T_{p_i}P(V) = \sum_{s=1}^{n} a_s t^{s-k}$.

Proposition 2.2 shows that for a linear action of $\mathbf{Z}_m$ on $P(\mathbf{C}^n)$ with odd m, the fixed point set consists of $P(\mathbf{C}^{n_i})$, $i = 1, \ldots, l$, and $\sum_{i=1}^{n} n_i = n$. Moreover it is not difficult to see that the inclusion $j_i : P(\mathbf{C}^{n_i}) \longrightarrow P(\mathbf{C}^n)$ induces isomorphisms $j_i^* : \mathrm{H}^*(P(\mathbf{C}^n); \mathbf{Z}) \longrightarrow \mathrm{H}^*(P(\mathbf{C}^{n_i}); \mathbf{Z})$ for $* \leq 2n_i - 2$.

Bredon and Su showed that any odd prime order cyclic group action on a manifold X which is mod p cohomology equivalent to $P(\mathbf{C}^n)$ resembles a linear action.

THEOREM 2.3. [Br]. *Let X be a smooth $\mathbf{Z}_p$ manifold with* mod p *cohomology of $P(\mathbf{C}^n)$. Let $F_1, \ldots, F_l$ be the fixed point components. Then each F_i is a smooth manifold with* mod p *cohomology $P(\mathbf{C}^{n_i})$, and the inclusion $j_i : F_i \longrightarrow X$ induces a surjection $\mathrm{H}^*(X ; \mathbf{Z}_p) \longrightarrow \mathrm{H}^*(F_i ; \mathbf{Z}_p)$ in the* mod p *cohomology. In addition $\sum_{i=1}^{l} n_i = n$.*

It is known that there are homotopy complex projective spaces with smooth $\mathbf{Z}_p$ actions such that for some component F of the fixed point set, $\mathrm{H}^*(F ; \mathbf{Z})$ contains torsion prime to p. Thus even though a smooth $\mathbf{Z}_p$ action on a mod p cohomology complex projective space resembles linear action when we consider mod p cohomology, But considering integral cohomology even a smooth $\mathbf{Z}_p$ action on a homotopy complex projective space does not resemble a linear action.

In transformation group theory we usually choose a model case action which is well understood and try to compare a given action with a model case action. In our situation a model case action is a linear action on a (standard) complex projective space.

Thus we may ask the following rather broad question: Suppose a group G acts smoothly on a homotopy complex projective space. How different is the action from a linear action?

In the following sections we will discuss more about comparisions between arbitrary actions on homotopy complex projective spaces and linear actions on standard complex projective spaces.

3. Background materials

In this section we discuss some basic materials such as G fiber homotopy equivalence, G transversality, G surgery, and Atiyah-Bott's G

signature formula, which will be used for later constructions of exotic actions on homotopy complex projective spaces.

In classical surgery theory we usually proceed as follows. We start with a fiber homotopy equivalence $\omega : \xi_+ \longrightarrow \xi_-$ between two vector bundles over a manifold M. We then show that ω is properly homotopic to a map $h : \xi_+ \longrightarrow \xi_-$ which is transverse to the zero section $M \subset \xi_-$. Let $X = h^{-1}(M)$ and $f = h|_X : X \longrightarrow M$. Then we have the data (X, f, b) called a normal map where $f : X \longrightarrow M$ is of degree 1, and $b : TX \longrightarrow f^*(TM + \xi_+ - \xi_-)$ is a stable vector bundle isomorphism.

We want to convert the normal map (X, f, b) via surgery into another normal map (X', f', b') such that f' is a homotopy equivalence. This is not always possible. That is, there is an obstruction $\sigma(f)$, which is an element of the Wall group $\mathrm{L}^h_{\dim M}(\mathbf{Z}[\pi_1(M)], \omega)$ such that $\sigma(f) = 0$ if and only if we can convert the normal map (X, f, b) into a homotopy equivalence.

If we want to do the above argument in G-category, with group actions, things are getting complex and sometimes we encounter with new obstructions. Petrie and Dovermann investigated the case and developed a theory on the subject, which we now review. A basic reference for this is [PR].

G fiber homotopy equivalence

Let $\xi = (E, p, X)$ and $\xi' = (E', p', X')$ be G bundles and let $f : X \longrightarrow X'$ be a G map. Let $u : \xi \longrightarrow \xi'$ and $v : \xi \longrightarrow \xi'$ be G bundle maps over f. We say u and v are G fiber homotopic over f if there is a G map $F : E \times I \longrightarrow E'$ such that $F(E_x, t) \subset E'_{f(x)}$ for all $x \in X, t \in I$ and $F(e, 0) = u(e)$, $F(e, 1) = v(e)$ for all $e \in E$. Here $E \times I$ is a G space with the trivial G action on I, and E_x and $E'_{f(x)}$ are fibers over x and $f(x)$ respectively. Let $\xi = (E, p, X)$ and $\xi' = (E', p', X)$ be two G bundles over X. A G bundle map $u : \xi \longrightarrow \xi'$ over 1_X is a G fiber homotopy equivalence if there is a G bundle map $u' : \xi' \longrightarrow \xi$ over 1_X such that $u'u$ and vv' are G fiber homotopic to 1_ξ and $1'_\xi$, respectively. ξ and ξ' are stably G fiber homotopy equivalent if there is a G bundle ζ over X such that $\xi \oplus \zeta$ and $\xi' \oplus \zeta$ are G fiber homotopy equivalent.

Let ξ and ξ' be G vector bundles over X. We say that ξ and ξ' are equivariant J equivalent if there is a G vector bundle ξ and a proper fiber preserving G map $w : \xi \oplus \zeta \longrightarrow \xi' \oplus \zeta$ such that $\deg \omega_x^H = \pm 1$ for all subgroups H of G and all $x \in X^H$.

Here is a theorem which relates stable G-fiber homotopy equivalences and equivariant J equivalences.

THEOREM 3.1. [PR]. *Let ξ and ξ' be G vector bundles over a compact G manifold X. Then ξ and ξ' are stably G-fiber homotopy equivalent iff ξ and η are equivariant J-equivalent.*

We now give an example of G fiber homotopy equivalence of G vector bundles over a complex projective space, which is important in later sections.

EXAMPLE 3.2. Let p and q be relative prime non-negative integers. Let α be any integer. Let G be a cyclic group of order m. Consider $G \times S^1$ representations

$$U = t_s^{\alpha p} + t_s^{\alpha q}, \quad V = t_s^{\alpha pq} + t_s^{\alpha}.$$

Here t_s^k is a complex 1 dimensional representation with the action of $G \times S^1$ defined by

$$\left(e^{\frac{2\pi si}{m}}, e^{i\theta}\right) \cdot z = e^{\frac{2\pi ski}{m}} \cdot e^{ik\theta} z$$

for $e^{i\theta} \in S^1 \subset \mathbf{C}$, $e^{\frac{2\pi si}{m}} \in G \subset \mathbf{C}$, and $z \in t^k = \mathbf{C}$. Choose natural numbers r and s such that

$$-rp + sq = 1.$$

Define a map $w : U \longrightarrow V$ by

$$\omega(z_1, z_2) = (\bar{z_1}^r z_2^s, z_1^p + z_2^q).$$

It is known that ω is a proper S^1 map of degree 1. Let A be a unitary representation of a group G. Then we have an S^1 equivariant map

$$Id \times \omega : S(A) \times U \longrightarrow S(A) \times V$$

where the S^1 action is defined by

$$(e^{i\theta})(x, z) = (xe^{i\theta}, e^{-i\theta} \cdot z)$$

for $e^{i\theta} \in S^1$, and $(x, z) \in S(A) \times U$ or $S(A) \times V$. Let $\widehat{U} = (S(A) \times U)/S^1$, let $\widehat{V} = (S(A) \times V)/S^1$ be orbit spaces, and let $\widehat{\omega} : \widehat{U} \longrightarrow \widehat{V}$ be the induced G fiber map between complex vector bundles $\widehat{U}$ and $\widehat{V}$ over the complex projective space $P(A)$. Let $x \in \mathrm{H}^2(P(A)\,; \mathbf{Z})$ be a generator.

LEMMA 3.3. *The total Pontrjagin classes for bundles $\widehat{U}$ and $\widehat{V}$ are*

$$p(\widehat{U}) = (1 + \alpha^2 p^2 x^2)(1 + \alpha^2 q^2 x^2)$$
$$p(\widehat{V}) = (1 + \alpha^2 p^2 q^2 x^2)(1 + \alpha^2 x^2)$$

Proof. Let γ be the canonical complex line bundle over $P(A)$. Forgetting the G actions on $\widehat{U}$ and $\widehat{V}$ we have $\widehat{U} = \gamma^{\alpha p} + \gamma^{\alpha q}$ and $\widehat{V} = \gamma^{\alpha} + \gamma^{\alpha p q}$ where γ^k is the k-fold tensor product of r. Since $p(r^k) = 1 + k^2 x^2$ we have the result.

If $\deg \widehat{\omega}^H_x = \pm 1$ for all subgroups H of G and all $x \in X^H$, then $\widehat{\omega} : \widehat{U} \longrightarrow \widehat{V}$ is a G fiber homotopy equivalence. In practice, we can achieve the condition that $\deg \widehat{w}^H_x = \pm 1$ for all subgroups H of G and all $x \in X^H$ easily by imposing some conditions on p, q and α. **G transversality** Suppose we have a proper G fiber map $\omega : \xi \longrightarrow \xi'$ between G vector bundles over X. When is ω properly G homotopic to $\omega' : \xi \longrightarrow \xi'$ where ω' is transversal to the zero section X of ξ' ? If G is trivial, due to Thom's transversality theory, this is always possible. But if G is not trivial, Petrie [PR] found out that there is an obstruction to achieving the above transversality. Here we only state his result on G-transversality for the case when G is finite.

For any irreducible representation χ of G d_χ denotes the real dimension of the division algebra $\mathrm{Hom}_G(\chi, \chi)$. Thus $d_\chi = 1, 2,$ or 4. Let $\omega : \xi \longrightarrow \xi'$ be as above. Let x be any irreducible representation of G_x for $x \in X$. Let $a_\chi(x)$ be the multiplicity of χ in $(T\xi)_x$ and $b_\chi(x)$ be the multiplicity of χ in $(\xi')_x$.

THEOREM 3.4 . (G TRANSVERSALITY THEOREM). *Under the above assumption ω is properly G homotopic to a G map h transverse to the zero section $X \subset \xi'$ if dim $F \leq d_x(a_\chi(x) - b_\chi(x) + 1) - 1$ for every point $x \in X$ and any χ such that $b_\chi(x) \neq 0$. Here F is a component of the G_x fixed point set of X.*

COROLLARY 3.5. *Suppose X^G consists of isolated fixed points and G acts semifreely (i.e. the G action on $X - X^G$ is free). Then ω is properly G homotopic to a G map h which is transversal to $X \subset \xi'$ if $\xi'|_x$ is a subrepresentation of $(TX \oplus \xi)_x$ as a representation of G_x for all $x \in X$.*

G-surgery theory

Let $f : X \longrightarrow Y$ be a G map between two G manifolds. G surgery theory is applied to an object called a G normal map, so that $f : X \longrightarrow Y$ is converted into a G homotopy equivalence $f' : X' \longrightarrow Y$. The definition of G normal map varies slightly depending on the surgery problem we want to solve. For our purpose we use the following definition. For simplicity, let's assume that each fixed point set X^K is connected.

DEFINITION 3.6.. *A G normal map $w = (X, f, b)$ consists of the following data:*

(1) *$f : X \longrightarrow Y$ is a G map between compact oriented G manifolds.*
(2) *$\deg f^K = 1$ for all $K \subset G$.*
(3) *There exists a vertual G vector bundle $\xi \in KO_G(Y)$ and a stable G bundle isomorphism $b : sTX \longrightarrow sf^*\xi$.*
(4) *Y^K is simply connected, and $\dim Y^K$ is either 0 or greater than 4.*
(5) *Y satisfies the gap hypothesis, i.e., for each component F of Y^K for $1 \neq K \subset G$, $2 \dim F < \dim Y$.*
(6) *X is a stable manifold (iff TX is a stable bundle), that is, for each $x \in X$ the G_x representation T_xX satisfies*

$$m_1(T_xX) \leq d_\chi m_\chi(T_xX) \quad \textit{if} \quad m_\chi(T_xX) \neq 0$$

for any irreducible representation χ of G_x. Here $m_\chi(T_xX)$ is the multiplicity of χ in T_xX.
(7) *$\mathrm{Iso}(X) = \mathrm{Iso}(Y)$.*

Let $\omega : \xi_+ \longrightarrow \xi_-$ be a G homotopy equivalence . From Theorem 3.4 we know when ω is properly G homotopic to $h : \xi_+ \longrightarrow \xi_-$ which is transversal to Y. Let $h^{-1}(Y) = X$ and $f = h|_X$. Then we have a G map $f : X \longrightarrow Y$ and a stable G vector bundle isomorphism $b : sTX \longrightarrow sf^*\xi$

where $\xi = \xi_+ - \xi_- \in KO_G(Y)$. The following theorem shows when this data (X, f, b) is a G normal map.

THEOREM 3.7. *With the above notation, suppose the following conditions:*

(1) *Y is a smooth compact oriented G manifold, Y^K is simply connected, and Y satisfies the gap hypothesis.*
(2) $\mathrm{Iso}(\xi_+) \subseteq \mathrm{Iso}(Y)$.
(3) *$TY + \xi$ is a stable bundle over Y.*
(4) *$\omega : \xi_+ \longrightarrow \xi_-$ satisfies the G transversality condition in Theorem 3.4.*

Then (X, f, b) is a G normal map.

For the proof see Theorem 11.22 of [PR].

It is well known that $f : X \longrightarrow Y$ is a G homotopy equivalence between G manifolds iff $f^K : X^K \longrightarrow Y^K$ is a homotopy equivalence for all $K \subset G$. Thus in order to apply G surgery to a G normal map $w = (X, f, b)$, so that we can achieve a G homotopy equivalence, we have to do surgery on each $w^K = (X^K, f^K, b^K)$ for all $K \subset G$ to get homotopy equivalences $f'^K : X'^K \longrightarrow Y^K$ for all $K \subset G$. This is not always possible. Namely, there is a series of obstructions $\{\sigma(f^K)\}$, one for each w^K.

Here each obstruction $\sigma(f^K)$ is an element of the Wall group $L_n^h(G/K, w)$. For the definition of Wall group see [Wa].

THEOREM 3.8[WA]. *Let $w = (X, f, b)$ be a G normal map. Then w can be converted to a G normal map $w' = (X', f', b')$ with a G homotopy equivalence $f' : X' \longrightarrow Y$ if and only if $\sigma(f^K)$ vanishe for all $K \subset G$.*

G-signature Formula

Here we review Atiyah-Bott G signature formula. We first give the definition of G signature. Let X be a $2n$ dimensional compact oriented G manifold. Suppose G preserves the orientation of X. Namely, if $[X] \in H_{2n}(X)$ is the fundamental class of X, then $g_*[X] = [X]$ for all $g \in G$. Define a bilinear form B on $H^n(X; \mathbf{R})$ by

$$B(x, y) = (x \cup y) \cap [X]$$

for $x, y \in H^n(X;\mathbf{R})$. Since $x \cup y = (-1)^{n^2}(y \cup x)$, B is symmetric if n is even and skew-symmetric if n is odd. $B(gx, gy) = B(x, y)$ and B is non-degenerate. Let $< \, , \, >$ be a G invariant positive definite inner product on $H^n(X;\mathbf{R})$. Define a linear map $A : H^n(X;\mathbf{R}) \longrightarrow H^n(X;\mathbf{R})$ by $B(x, y) =< x, Ay >$. Then A is equivariant and $A^* = (-1)^{n^2}A$.

CASE 1: n is even

Then $A^* = A$. All eigenvalues of A are real. Let H^+(resp. H^-) be positive (resp. negative) eigen space of A. These eigen spaces are G invariant. Thus $H^{\pm}$ are G representations. Let $\rho^{\pm}$ be the corresponding characters. Thus $\rho^+, \rho^- \in RO(G) \subset R(G)$. The G-signature of X $\mathrm{Sign}(G, X) = \rho^+ - \rho^- \in RO(G)$. It is easy to show that $\mathrm{Sign}(G, X)$ is independent of the choice of $< \, , \, >$.

CASE 2: n is odd

If A is skew adjoint, let $J = A(AA^*)^{-\frac{1}{2}}$, where $(AA^*)^{1/2}$ is the positive square root AA^*. Then $J^2 = -I$, and we can regard $H^n(X;\mathbf{R})$ as a $\mathbf{C}$-module , where $u + iv$ acts on $w \in H^n(X;\mathbf{R})$ by $(u + iv) \cdot w = u - w + J(v \cdot w)$. This commutes with the G action. Thus we obtain a character $\rho \in R(G)$. Again, ρ is independent of the choice of $< \, , \, >$, and define $\mathrm{Sign}(G, X) = \rho - \rho^* \in R(G)$.

The evaluation $\mathrm{Sign}(G, X)(g)$ at $g \in G$ is denoted by $\mathrm{Sign}(g, X)$. If $\langle g \rangle$ denotes the subgroup of G generated by g, then $\mathrm{Sign}(g, X) = \mathrm{Sign}(\langle g \rangle, X)(g)$. From the definition G signature is a global property of the group action on a manifold. But we can relate this global information to a local behavior of the action. Namely Atiyah and Bott [AB] find the following theorem.

THEOREM 3.9. (G-SIGNATURE FORMULA).

$$\mathrm{Sign}(g, X) = \sum_{F \subset X^g} L(TF) \cdot \mathcal{L}(v(F, X))[F].$$

Here L is the L-polynomial of Pontrjagin classes of TF. F is a component of the fixed point set X^g. It should be pointed out that a reversal of the orientation on a neighborhood of F will cause the change of sign for the contribution of F in Theorem 3.9.

We are going to use the following proposition which follows easily from the G signature formula. Suppose N is a neighborhood of the component F of X^g in X.

PROPOSITION 3.10. *X^g consists of two unions of components B_1 and B_2 with neighborhoods N_1 and N_2. Suppose there is an orientation reversing equivariant diffeomorphism from N_1 to N_2. Then* $\mathrm{Sign}(g, X) = 0$.

Proof.

$$\mathrm{Sign}(g, X) = L(B_1)\mathcal{L}(\nu(B_1, X))[B_1] + L(B_2)\mathcal{L}(\nu(B_2, X))[B_2].$$

Since there is an orientation reversing G diffeomorphism

$$L(B_1)\mathcal{L}(\nu(B_1, X))[B_1] = -L(B_2)\mathcal{L}(\nu(B_2, X))[B_2].$$

In stead of giving a precise explanation of $\mathcal{L}(\nu(F, X))$, we will give explicit formulas for some group actions.

Suppose X^g consists of isolated points. Since dim X is even, $T_pX = E_1 \oplus \cdots \oplus E_n$ for $p \in X^g$ where E_k is a 2-plane where g acts on E_i by rotation through an angle θ_k. Define

$$\nu(p) = \prod_{k=1}^{n} \frac{e^{-i\theta_k} - e^{i\theta_k}}{(1 - e^{-i\theta_k})(1 - e^{i\theta_k})} = \prod_{k=1}^{n} \coth(\frac{i\theta_k}{2}).$$

COROLLARY 3.11. *Under the above assumption*

$$\mathrm{Sign}(g, X) = \sum_{p \in X^g} \nu(p).$$

Suppose X^g consists of a 2n-2 dimensional component F and an isolated point p. In this case the normal bundle $\nu(F, X)$ has a complex line bundle structure, and let its total Chern class be $1 + c_1(\nu) \in \mathrm{H}^*(F\,; \mathbf{Z})$. Due to the choice of complex structure $c_1(\nu)$ is defined up to sign.

Let $0 < \theta < \pi$ and let ν be any complex line bundle over F write

$$\coth(x + i\theta/2) = a_0(1 + \sum_{j=1}^{\infty} a_j x^j).$$

Let $\mathcal{L}_\theta(\nu) = a_0(1+\sum_{j=1}^{\infty} a_j c_1(\nu)^j) \in \mathrm{H}^*(F\,;\,\mathbf{Z})$. Let $L(F)$ be the Hirzebruch's L-polynomial.

$$L(F) = 1 + \frac{1}{3}p_1(F) + \frac{1}{45}(792p_2(F) - p_1(F)^2) + \cdots \in \mathrm{H}^*(F\,;\,\mathbf{Z}).$$

Here $p_i(F)$ is the i-th Pontrjagin class of F. Suppose $g \in G$ is of order p.
Let $\theta = 2\pi/p$.

COROLLARY 3.12. *Under the above assumption* $\mathrm{Sign}(g,x) = \pm L(F)\cdot \mathcal{L}_\theta(\nu(F,X)) \pm \nu(p)$. *The signs depend on choices of orientations and complex stuctures.*

G-surgery obstructions

We know that surgery obstructions are elements of the Wall groups $L_n^h(\mathbf{Z}[G/K],\omega)$. On the other hand it is a fact that surgery obstructions to achieving G simple homotopy equivalence are elements of the Wall groups $L_n^s(\mathbf{Z}[G/K],\omega)$.

There is a signature homomorphism

$$\mathrm{Sign} : L_n^{h,s}(\mathbf{Z}[G/K],1) \longrightarrow \mathrm{R}(G)$$

which is defined as follows: If $w = (X,f,b)$ is a G normal map and $\sigma(f^K) \in L_n^{h,s}(\mathbf{Z}[G/K],1)$ is a surgery obstruction for even n, then $\mathrm{Sign}(G/K,\sigma(f^K)) = \mathrm{Sign}(G/K,Y^K) - \mathrm{Sign}(G/K,X^K)$.

THEOREM 3.13. [WA]. *If G is an odd order cyclic groups,*

$$\ker(\mathrm{Sign} : L_{2k}^s(\mathbf{Z}[G],1) \longrightarrow \mathrm{R}(G)) = \begin{cases} 0 & \text{if } k \equiv 0(2) \\ \mathbf{Z}_2 & \text{if } k \equiv 1(2) \end{cases}$$

and the element in $\mathbf{Z}_2(k \equiv 1(2))$ is detected by the Arf invariant of the normal map forgetting the group action.

The Rothenberg sequence relates the obstruction groups L^h and L^s by the long exact sequence

$$(3.14) \qquad H^{k+1}(\mathbf{Z}_2, Wh(G)) \longrightarrow L_K^s(\mathbf{Z}[G],1)$$
$$\xrightarrow{j} L_K^h(\mathbf{Z}[G],1) \xrightarrow{\alpha} H^k(\mathbf{Z}_2, Wh(G))$$

4. Actions with isolated fixed points

Suppose a group G acts smoothly on a homotopy complex projective space X^{2d} with isolated fixed points. Then by theorem 2.3 the number of fixed points are $d+1$.

EXAMPLE 4.1. Let G be the cyclic group of order m. Let $A = \sum_{i=1}^{m} t^{a_i}$ where $a_i \not\equiv a_j(m)$ if $i \neq j$. Let $X = P(A)$. Then G acts on X with isolated fixed points $\{p_i : p_i = [0 : \cdots 0 : 1 : 0 \cdots 0]$, 1 in i-th place $1 \leq i \leq d+1\}$. Moreover at each p_k the isotropy representation

$$T_{p_k}X = \sum_{i \neq k} t^{a_i - a_k}.$$

Our main interest in this section is as follows: which homotopy complex projective spaces admit group actions with isolated fixed points? If so, how different are they from linear actions? We will answer these questions for cyclic groups of odd order.

The first construction of cyclic group of odd prime order actions on nonstandard homotopy complex projective spaces was done by Masuda and Tsai [MT].

THEOREM 4.1. [MT]. *Let m be an odd prime number and $2d \mid m-1$ for some integer $d \geq 2$. Then there are infinitely many homotopy complex projective spaces X^{4d-2} such that $G = \mathbf{Z}_m$ acts on X with $2d$ isolated fixed points.*

For a G representation V let $\lambda_{-1}(V) = \sum(-1)^i \lambda^i(V)$ be the Euler class of V. In the above theorem X is G homotopy equivalent to $P(A)$ for some G representation A. Let $\{p_i\}$ denote the isolated fixed points of both X and $P(A)$. Then the expression

$$\psi_i(t) = \lambda_{-1}(T_{p_i}P(A))/\lambda_{-1}(T_{p_i}X)$$

can be regarded as quantities which describes to what extent the action on X resembles a linear action on $P(\mathbf{C}^n)$. In the above theorem $\Psi_i(t) \neq \pm\Psi_j(t)$.

Tsai [Ts] improved the above result to cyclic groups of odd order, not necessarily prime.

Dovermann and Masuda [DM] improved the above result in some cases. The following theorem is one of them.

THEOREM 4.2. [DM]. *Let p and q be relatively prime integers, and let m be a natural number prime to $2pq(p^2-q^2)$. Let X be a homotopy $P(\mathbf{C}^4)$ with the first Pontrjagin class $p_1(X)=\{4-4(p^2-1)(q^2-1)\}x^2$ where x generates $H^2(X)$. Then X admits semifree effective smooth $\mathbf{Z}_m$ action with 4 isolated fixed points.*

Dovermann informed the auther recently that he could extend Theorem 4.2 to homotopy $P(\mathbf{C}^7)$. So far all results are about odd order cyclic group actions on homotopy $P(\mathbf{C}^n)$ such that $n\equiv 0(2)$.

The only result about cyclic group actions on homotopy $P(\mathbf{C}^n)$, odd n, with isolated fixed points is given by Tsai [Ts].

THEOREM 4.3. [Ts]. *Suppose G is a cyclic group of order m. If $2d \mid p_i - 1$ for each prime factor p_i of m, k is even, and the condition (*) is satisfied, then there is a non-standard homotopy $P(\mathbf{C}^{2d+1})X$ and smooth G action on X with isolated fixed points.*

Because of some technical reason the condition ($*$) will be stated later.

Tsai also gave an example of G action on nonstandard homotopy $P(\mathbf{C}^5)$ with isolated fixed points.

We now give ideas of proofs of the above theorems.

Proof of Theorem 4.1.

Let G^* denote the multiplicative group of units of G. If $2d \mid m-1$, there is $a\in G^*$ with order $2d$ in G^*. Now for $r\neq r'$ and $1\le r, r'\le 2d-1$ let

$$\begin{aligned}
A &= t + t^a + t^{a^2} + \cdots + t^{a^{2d-1}}\\
U &= t_s^{rp} + t_s^{rq}\\
V &= t_s^{r} + t_s^{rpq}\\
(p,q) &= 1\\
r &\equiv \pm(a^r - 1) \pmod m\\
rpq &\equiv \pm(a^{r'} - 1) \pmod m.
\end{aligned}$$

We now use the G fiber homotopy equivalence construction of Example 3.2 to get a G fiber homotopy equivalence $\widehat{\omega}: \widehat{U} \longrightarrow \widehat{V}$ over $P(A)$. Note that

$$\begin{aligned} P(A)^G &= \{p_i = [0:\cdots:0:1:0:\cdots:0] : i = 1,\ldots,2d\} \\ TP(A)|_{p_i} &= \sum_{j=1}^{2d-1} t^{a^i(a^j-1)} \\ \widehat{U}|_{p_i} &= t^{a^i rp} + t^{a^i rq} \\ \widehat{V}|_{p_i} &= t^{a^i r} + t^{a^i rpq} = t^{\pm a^i(a^r-1)} + t^{\pm a^i(a^{r'}-1)} \end{aligned}$$

Since $t^{a^i(a^r-1)} + t^{a^i(a^{r'}-1)}$ is a subrepresentation of $\sum_{j=1}^{2d-1} t^{a^i(a^j-1)}$, by Corollary 3.5 $\widehat{\omega}$ is properly G homotopic to $h: \widehat{U} \longrightarrow \widehat{V}$ such that h is transverse to $P(A)$. Let $X = h^{-1}(P(A))$, and let $f = h|_X$. We have $f: X \longrightarrow P(A)$ and a stable G vector bundle isomorphism $b: TX \longrightarrow f^*(TP(A) + \widehat{U} - \widehat{V})$. It is easy to see that such constructed data (X, f, b) satisfies the hypothesis of Theorem 3.5. Thus (X, f, b) is a G normal map. Since $\widehat{\omega}^G : \widehat{U}^G \longrightarrow \widehat{V}^G$ is bijective and h is obtained via G homotopy we may assume that $f^G : X^G \longrightarrow P(A)^G$ is bijective. Thus the only obstruction to achieving G homotopy equivalence via G surgery is $\sigma(f) \in L^h_{4d-2}(\mathbf{Z}[G], \omega)$. Actually $\sigma(f) \in L^h_{4d-2}(\mathbf{Z}[G], 1)$ because G preserves the orientation of $P(A)$.

LEMMA 4.4. *The above $\sigma(f)$ vanishes, if r is even.*

Proof. Consider the Rothenberg sequence 3.14

$$\begin{aligned} &\longrightarrow H^{4d-1}(\mathbf{Z}_2;\, Wh(G)) \longrightarrow L^s_{4d-2}(G,1) \\ &\longrightarrow L^h_{4d-2}(G,1) \xrightarrow{\alpha} H^{4d-2}(\mathbf{Z}_2;\, Wh(G)). \end{aligned}$$

It is known that $Wh(G)$ has no torsion. Thus $H^{4d-1}(\mathbf{Z}_2;\, Wh(G)) = 0$. On the otherhand the result of Dovermann and Rothenberg [DR] tells us that $\alpha(\sigma(f)) = 0$ because both $T_{p_i}X$ and $T_{p_{i+d}}X$ are equivalent as real representations. This means that there is a unique element $\sigma_s(f) \in L^s_{4d-2}(\mathbf{Z}(G), 1)$ which maps to $\sigma(f)$. For $g \in G - 1$ let us calculate

$$\mathrm{Sign}(\sigma_s(f))(g) = \mathrm{Sign}(g, X) - \mathrm{Sign}(g, P(A)).$$

$\text{Sign}(g, P(A)) = 0$ from the definition. $\text{Sign}(g, X) = \sum_{p_i \in X^g = X^G} \nu(p_i)$ by Corollary 3.11. But $|\nu(p_i)| = |\nu(p_{i+d})|$ because $T_{p_i}X$ and $T_{p_i+d}X$ are equivalent as real representations. Thus with an appropriate orientation $\text{Sign}(g, X) = 0$. Thus $\text{Sign}(\sigma_s(f))(g) = 0$ for all $g \in G - 1$. Theorem 3.13 says that $\sigma_s(f) \in \mathbf{Z}_2$ is detected by the Arf invariant. Lemma 3.2 of [MT] shows that the Arf invariant of $\sigma_s(f)$ is zero if r is even.

We thus have proved the following theorem which is a sharper statement than Theorem 4.1.

THEOREM 4.5. [MT]. *Let G be a cyclic group of odd prime order m. Let d be a natural number such that $2d \mid m - 1$. Then there exists a homotopy complex projective space $X(r,p,q)$ together with a G homotopy equivalence $f : X(r,p,q) \longrightarrow P(A)$ such that*

(1) *$TX(r,p,q)$ is stably isomorphic to $f^*(TP(A) + \widehat{U} - \widehat{V})$ as a G vector bundle.*
(2) *$X(r,p,q)^G$ consists of 2d points.*
(3) *$\Psi_i(t) = (1 - t^{a^{ir}})(1 - t^{a^{irpq}})/(1 - t^{a^{irq}})(1 - t^{a^{irp}})$ where p, q, r are defined as above, and r is even.*
(4) *The total Pontrjagin class of $X(r,p,q)$ is*

$$p(X) = (1 + x^2)^{2d}(1 + r^2p^2x^2)(1 + r^2q^2x^2)(1 + r^2x^2)^{-1}(1 + r^2p^2q^2x^2)^{-1}$$

Proof of theorem 4.2. The basic idea of the proof is the same as the proof of theorem 4.2. But here we have to make more careful choices of representations A, U, and V. Let positive integers p, q, and m be given such that

(1) (p, q) = 1 and $(m, 2pq(p^2 - q^2)) = 1$
set
(2) $\alpha = 2, \quad a_1 = p, \quad a_2 = q$
(3) $\beta = a_2 - a_1$
(4) $r = -a_1 - a_2$
choose integers c, d, e, f such that
(5) $re \equiv \alpha pq^2 \pmod{m}$
(6) $\beta c \equiv -\alpha p^2 q \pmod{m}$
(7) $ref \equiv \alpha pq \pmod{m}$
(8) $\beta cd \equiv rf \pmod{m}$

obviously

(9) $(m, c\,d\,e\,f\,\beta r) = 1$

By Dirichlet's theorem (each nonzero congruence class contains infinitely many primes) we may choose c, d, e, f such that

(10) $(c, d) = (e, f) = 1$

Let $A = t^{a_1} + t^{-a_1} + t^{a_2} + t^{-a_2}$. Then since $(a_1 \pm a_2, m) = (2a_1, m) = (2a_2, m) = 1$, $G = \mathbf{Z}_m$ acts semifreely with 4 isolated fixed points $p_1 = [1:0:0:0], \bar{p_1} = [0:1:0:0], p_2 = [0:0:1:0]$, and $\bar{p_2} = [0:0:0:1]$.

Let $U = t_s^{\alpha p} + t_s^{\alpha q}$ and $V = t_s^{\alpha} + t_s^{\alpha pq}$ be two S^1 representations. Let $B = t^{\beta c} + t^{\beta d} + t^{re} + t^{rf}$ and $C = t^{\beta} + t^{\beta cd} + t^{r} + t^{ref}$ be two G representations. As in Example 3.2 there is a proper G fiber homotopy equivalence $\widehat{\omega} : \widehat{U}_+ \longrightarrow \widehat{U}_-$ between vector bundles over $P(A)$. Let $\underline{B} = B \times P(A)$ and $\underline{C} = C \times P(A)$ be the trivial G vector bundles over $P(A)$. As in Example 3.2 there is a proper G map $\omega' : B \longrightarrow C$ of degree 1. Let $\underline{\omega}' : \underline{B} \longrightarrow \underline{C}$ be the induced G fiber homotopy equivalence. Set $\xi_+ = \widehat{U} \oplus \underline{B}$ and $\xi_- = \widehat{V} \oplus \underline{C}$ Let $\omega = \widehat{\omega} \oplus \underline{\omega}' : \xi_+ \longrightarrow \xi_-$ which is a G fiber homotopy equivalence. Because of the careful choice of 4.6 it is not difficult to that

$$\xi_-|_{p_i} = \widehat{V}|_{p_i} \oplus C = [t^{a_i \alpha pq} + t^{a_i \alpha}] + [t^{\beta cd} + t^{\beta} + t^{ref} + t^{r}]$$

and

$$\xi_-|_{\bar{p}_i} = \widehat{V}|_{\bar{p}_i} \oplus C = [t^{-a_i \alpha pq} + t^{-a_i \alpha}] + [t^{\beta cd} + t^{\beta} + t^{ref} + t^{r}]$$

(as real representations) are subrepresentations of

$$\begin{aligned} T\xi_+|_{p_i} &= T_{p_i}(P(A)) \oplus \widehat{U}|_{p_i} \oplus B \\ &= [t^{-2a_i} + t^{a_j - a_i} + t^{-a_j - a_i}] + [t^{a_i \alpha p} + t^{a_i \alpha q}] \\ &\quad + [t^{\beta c} + t^{\beta d} + t^{re} + t^{rf}] \end{aligned}$$

and

$$\begin{aligned} T\xi_+|_{\bar{p}_i} &= T_{\bar{p}_i}(P(A)) \oplus \widehat{U}|_{\bar{p}_i} \oplus B \\ &= [t^{2a_i} + t^{a_j + a_i} + t^{-a_j + a_i}] + [t^{-a_i \alpha p} + t^{-a_i \alpha q}] \\ &\quad + [t^{\beta c} + t^{\beta \alpha} + t^{re} + t^{rf}] \end{aligned}$$

respectively. Thus the G fiber homotopy equivalence $\omega : \xi_+ \longrightarrow \xi_-$ satisfies the condition of the G transversality Theorem 3.4. Therefore ω is G homotopic to h which is transverse to $P(A)$. As in the proof of Theorem 4.1 this defines a G normal map (X, f, b) with the target space $P(A)$. Moreover $f^G : X^G \longrightarrow P(A)^G$ is a bijection. There is a surgery obstruction $\sigma(f) \in L_6^h(\mathbf{Z}[G], 1)$. Consider the Rothenberg sequence

$$0 \longrightarrow L_6^s(\mathbf{Z}[G], 1) \longrightarrow L_6^h(\mathbf{Z}[G]; 1) \xrightarrow{\alpha} H^6(\mathbf{Z}_2 ; Wh(G)) \longrightarrow$$

Since $T_{p_i}P(A)$ and $T_{\bar{p}_i}P(A)$ are equivalent as real representations $\alpha(\sigma(f)) = 0$ by the result of [DR]. Since $\alpha(\sigma(f)) = 0$ there exists unique $\sigma_s(f) \in L_6^s(\mathbf{Z}(G), 1)$ which maps to $\sigma(f)$. Since there exists orientation reversing equivariant diffeomorphism from a neighborhood N_i of p_i to a neighborhood $\bar{N}_i$ of $\bar{p}_i$, by Proposition 3.10 $\mathrm{Sign}(g, X) = 0$. Then Lemma 4.4 implies that $\sigma(f) = 0$.

Sketch of Proof of Theorem 4.3.

For simplicity we shall prove the theorem only for odd prime order cyclic group case. As in the proof of theorem 4.1 choose $a \in G^*$ with order $2d$ in G^* such that $2d \mid m - 1$. Let

$$A = 1 + t^a + t^{a^2} + \cdots t^{a^{2d-1}}$$

Since $a^i \not\equiv 0 \pmod m$ for any i and $a^i \not\equiv a^j \pmod m$ for $i \neq j$ and $0 \leq i, j \leq 2d - 1$, $P(A)^G$ consists of isolated fixed points. Suppose we can find a proper G fiber homotopy equivalence $\omega : \xi_+ \longrightarrow \xi_-$ of G vector bundles over $P(A)$ such that $(TP(A) + \xi_+ - \xi_-)|_{P(A)^G} = TP(A)|_{P(A)^G}$. Also assume the following condition $(*)$.

CONDITION $(*)$.: There exists a G vertual vector bundle $\mu = \mu_+ - \mu_- \in KO_G(P(A))$ such that

(1) there exists a G fiber homotopy equivalence

$$\varepsilon : \mu_+ \longrightarrow \mu_-$$

(2) $\mu|_{p_i} = \mu_+|_{p_i} - \mu_-|_{p_i} = 0$ as a real representation of G at each fixed points $p_i \in P(A)^G$. Furthermore, as a complex representation , if $t^a - t^{-a}$ is a summand of $\mu|_{p_i}$, then there are even copies of $t^a - t^{-a}$ in $\mu|_{p_i}$ for each i.

(3) $< L(TP(A) + \xi + \mu), [P(A)] > = 1$.

We have a G fiber homotopy equivalence $w' = w \oplus \varepsilon : (\xi_+ \oplus \mu_+) \longrightarrow (\xi_- \oplus \mu_-)$ satisfying the G transversality condition because $(TP(A) + \xi_+ - \xi_-)|_{P(A)^G} = TP(A)|_{P(A)^G}$ and $\mu_+|_{p_i} = \mu_-|_{p_i}$ as real representations of G for all $p_i \in P(A)^G$. Thus we can find a G normal map $w = (X, f, b)$ with $f : X \longrightarrow P(A)$ and $b = TX \longrightarrow f^*(TP(A) + \xi + \mu)$ where $\xi = \xi_+ - \xi_-$ and $\mu = \mu_+ - \mu_-$. As before we may assume that f^G is a bijection. So the only surgery obstruction to G homotopy equivalence is $\sigma(f) \in L^h_{4n}(\mathbf{Z}[G], 1)$. Since $T(P(A) + \xi)|_{P(A)^G} = TP(A)|_{P(A)^G}$ and $\mu|_{p_i} = 0$, $\alpha(\sigma(f)) = 0$ where α is the map in the Rothemberg sequence 3.14, and $\mathrm{Sign}(\sigma(f))(g) = 0$ for $g \neq 1$. If $g = 1$, then $\mathrm{Sign}(\sigma(f))(1) = \langle L(TX), [X]\rangle - \langle L(TP(A), [P(A)]\rangle = \langle L(TP(A) + \xi + \mu), [P(A)]\rangle - 1 = 0$.

It remains to prove that there exists a proper G fiber homotopy equivalence $\omega : \xi_+ \longrightarrow \xi_-$ of G vector bundles over $P(A)$ such that $(TP(A) + \xi)|_{P(A)^G} = TP(A)|_{P(A)^G}$.

This can be done as follows. Let $q_0, \ldots, q_{d-1}, p_0, \ldots, p_{2d-1}$ be pairwise relative prime integers such that

$$\begin{aligned} p_0 &\equiv -1 \pmod m \\ p_i &\equiv a^i - 1 \pmod m \qquad i = 1, 2, \ldots, d-1 \\ q_j p_{k+j} &\equiv a^{k+j} - 1 \pmod m \qquad j = 0, 1, \ldots, d-1 \\ q_j &\equiv \beta \pmod m \quad \text{for all } j \end{aligned}$$

where $\beta \in G^*$ of order 4 and $a \in G^*$ of order 2d. Let

$$U = \sum_{j=0}^{d-1} t_s^{p_j} + \sum_{j=0}^{d-1} t_s^{q_j + p_{d+j}}$$

$$V = \sum_{j=0}^{d-1} t_s^{q_j p_j} + \sum_{j=0}^{d-1} t_s^{-p_{d+j}}$$

Let

$$\begin{aligned} U_0 &= U, \\ U_1 &= t_s + t_s^{p_0 p_1} + \sum_{j=2}^{d-1} t_s^{p_j} + \sum_{j=0}^{d-1} t_s^{q_j + p_{d+j}}, \ldots, U_{2k-1} \\ &= (2d-1) t_s + t_s^{\Pi p_j \Pi q_j}. \end{aligned}$$

Then there exists a proper G fiber homotopy equivalence $\omega_r : \widehat{U}_r \longrightarrow \widehat{U}_{r+1}$. Thus the composition $\omega_+ = \omega_{2d-2} \circ \cdots \circ \omega_0 : \widehat{U}_0 \longrightarrow \widehat{U}_{2d-1}$ is a proper G fiber homotopy equvialence. Similarly there is a proper G fiber homotopy equivalence

$$\omega_- : \widehat{V}_0 \longrightarrow \widehat{V}_{2d-1}.$$

But $\widehat{U}_{2d-1} = \widehat{V}_{2d-1}$. Therefore there is a proper G fiber homotopy equivalence $\omega : \widehat{U} \longrightarrow \widehat{V}$. Let $\xi_+ = \widehat{U}$ and $\xi_- = \widehat{V}$. Then it is an elementary check to see that

$$(TP(A) + \xi)|_{P(A)^G} = TP(A)|_{P(A)^G}.$$

Note that the isotropy representations $TX|_{P(A)^G}$ are the same as those of a linear action's.

5. Rigid Versus Nonrigid Actions

In this section we summarize results on cyclic group actions of type II_0 on homotopy complex projective space X i.e., X^G consists of a codimension 2 manifold F and an isolated point P. The following example shows a type II_0 linear action.

EXAMPLE 5.1. Let $G = \mathbf{Z}_m$.

$$A = nt^a + t^b$$

where $(a - b, m) = 1$, $a \not\equiv b \pmod m$. Then $P(A)$ admits a type II_0 action of G such that

$$P(A)^G = P(nt^a) \coprod P(t^b).$$

Note that$P(nt^a)$ is $\mathbf{C}P^{n-1}$ and $P(t^b)$ is a point. Here $\coprod$ means disjoint union.

From now on until the end of this section X is a cohomology $\mathbf{C}P^n$ and G is a cyclic group of odd prime order p. Let G acts smoothly on X with a type II_0 action with $X^G = F \amalg P$. In this section we use the notation $\mathbf{C}P^n$ in stead of $P(\mathbf{C}^{n+1})$. Let $j : F \hookrightarrow X$ be the embedding. By Theorem 2.3 F is a mod p cohomology $\mathbf{C}P^{n-1}$. Since X is a cohomology $\mathbf{C}P^n$

$$\mathrm{H}^*(X\,;\,\mathbf{Z}) \cong \mathbf{Z}[x]/x^{n+1}$$

where $x \in \mathrm{H}^2(X\,;\,\mathbf{Z})$ is a generator. Set

$$D_X(F) = \left|\left(j^*(x)\right)^{n-1}[F]\right|.$$

This integer $D_X(F)$ is called the defect of F in X.

DEFINITION 5.2. *A type* II_0 *action of* $G = \mathbf{Z}_p$ *on a cohomology* $\mathbf{C}P^n$ X *is called algebraically standard or rigid if the following conditions are satisfied:*

(1) *The total Pontrjagin class is*

$$p(F) \equiv (1 + j^*x^2)^n \quad (\text{mod } torsion) \text{ in } \mathrm{H}^*(F\,;\,\mathbf{Z}).$$

(2) *The defect is* $D_X(F) = 1$.

(3) *The isotropy representation at the isolated fixed point* $T_P X = n\nu_x(F, X)$ *where* $\nu_x(F, X)$ *is the normal representation of* F *in* X *at any point* $x \in F$.

LEMMA 5.3. *Algebraically standard type* II_0 *action satisfies the following:*

(1) $p(X) = (1 + x^2)^{n+1}$

(2) $\mathbf{Z}[x_0]/x_0^n \longrightarrow \mathrm{H}^*(F\,;\,\mathbf{Z})/\text{torsion}$ *is an isomorphism where* $x_0 = j^*x$.

(3) $c_1(\nu(F, x)) \equiv \pm x_0$ (mod torsion).

Proof. Here we only prove (2) and (3).
(1) requires some involved work and we refer the reader to [Ma]. Let $j : F \hookrightarrow X$ be the inclusion. Define the Gysin map $j_! = (\cap[X])^{-1} j^*(\cap F)$ so that we have the commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^k(F\,;\,\mathbf{Z}) & \xrightarrow{\;j_!\;} & \mathrm{H}^{k+2}(X\,;\,\mathbf{Z}) \\
\cong \big\downarrow \cap[F] & & \cong \big\downarrow \cap[X] \\
\mathrm{H}_{2n-2-k}(F\,;\,\mathbf{Z}) & \xrightarrow{\;j^*\;} & \mathrm{H}_{2n-2-k}(X\,;\,\mathbf{Z})
\end{array}$$

The Gysin map has the following properties:

(a) $j^*j_!(1_F) = e(\nu)c_1(\nu)$, where $1_F \in H^0(F;\mathbf{Z})$ is a generator.

(b) $j_!(1) = D_X(F)\cdot x$

(a) follows from the following commutative diagram

$$
\begin{array}{ccccccccc}
\mathrm{H}^0(F) & \xrightarrow{PD} & \mathrm{H}_{2n-2}(F) & \xrightarrow{j_*} & \mathrm{H}_{2n-2}(X) & \xrightarrow{PD^{-1}} & \mathrm{H}^2(X) & \xrightarrow{j^*} & \mathrm{H}^2(F) \\
\Big\| & & \Big\downarrow \approx & & \Big\downarrow & & & & \Big\| \\
\Big\| & & & & \mathrm{H}_{2n-2}(X, X-F) & & \xrightarrow{PD^{-1}} & & \mathrm{H}^2(F) \\
\Big\| & & & & \Big\downarrow \approx \text{excision} & & & & \Big\uparrow PD^{-1} \\
\Big\| & & \mathrm{H}_{2n-2}(\nu) & \longrightarrow & \mathrm{H}_{2n-2}(\nu,\dot\nu) & & \xrightarrow{Thom.isom.} & & \mathrm{H}_{2n-4}(F) \\
\Big\| & & \Big\uparrow PD & & \Big\uparrow PD & & & & \Big\uparrow PD \\
\mathrm{H}^0(F) & \xrightarrow{Thom.isom.} & \mathrm{H}^2(\nu,\dot\nu) & \longrightarrow & \mathrm{H}^2(\nu) & & \xrightarrow{\cong} & & \mathrm{H}^2(F)
\end{array}
$$

(b) follows from the following formula

$$j_!(u \cup j^*\nu) = j_!(u) \cup u \text{ for } u \in H^*(F) \text{ and } \nu \in H^*(X).$$

Indeed,

$$t^{n-1}j_!(1) = j_!(j^*t^{n-1}) = (D_X(F)\cdot t)\cdot t^{n-1}$$

Hence $j_!(1) = D_x(F)\cdot x$. The above formula follows from an elementary calculation. $c_1(\nu) = j^*j_!(1_F) = j^*(D_X(F)\cdot x) \neq 0$, and $j^*(x) \neq 0$ by Theorem 2.3. Since $D_X(F) = 1$ (3) follows. Moreover $j^*(x^{n-1})[F] = D_x(F) = 1$ implies that $j^*(x^{n-1})$ generates $H^{2n-2}(F;\mathbf{Z})$. This implies that $j^*(x)$ generates $H^*(F;\mathbf{Z})/$ torsion.

The notion of rigid action is first introduced by Dovermann in [D1], where he proved rigidity of cyclic group of odd order actions on low dimensional cohomology $\mathbf{C}P^n$.

THEOREM 5.4. [D1]. *Suppose X is a cohomology $\mathbf{C}P^n$ with a smooth type II_0 action of $G = \mathbf{Z}_p$ where p is an odd prime.*

(1) *If $n = 1, 2, 3$, the action is rigid for all p.*
(2) *If $n = 4$ and $p \geq 5$, the action is rigid.*
(3) *If $n = 5$ and*
  α) *$p \geq 59$ and $h_1(p) \equiv 1(2)$*
  (where h_1 is the relative class number of $\mathbf{Z}[\exp(2\pi i/p)]$) or
  β) *$7 \leq p \leq 53$, $p \neq 31$,*

then the action is rigid.

This theorem gives a positive answer to $\mathbf{Z}_p$ version of Petrie's conjecture for low dimensional $\mathbf{C}P^n$. Keeping Theorem 5.4. in mind, one might ask the following question:

QUESTION 5.5. *Does there exist a natural number $f(n)$ which depends only on n such that any smooth type II_0 action of $G = \mathbf{Z}_p$ with prime $p \geq f(n)$, is algebraically standard?*

Further computer assisted computation in the direction of Theorem 5.4. shows some evidence for positive answer to the above question. Namely, if $n = 6$ and $7 \leq p \leq 43$, or $n = 7$ and $11 \leq p \leq 19$, then the type II_0 $\mathbf{Z}_p$ action on cohomology $\mathbf{C}P^n$ is rigid [DMS]. The upper limit on p is given not because of some theoretic reasons but because of computer CPU time restriction. The following theorem gives a partial answer to 5.5.

For cohomology $\mathbf{C}P^n$ X, define a sequence of numbers $\bar{a}(X) = \{a_{4k}(X)\}$ such that the total Pontrjagin class $p(X) = \sum a_{4k}(X)x^{2k}$ where x is a generator of $\mathrm{H}^*(X\,;\,\mathbf{Z})$.

THEOREM 5.6. [DMS]. *Let $\{a_{4k}\} = \overline{a}$ be a sequence of integers. There exists a constant $c(\overline{a})$ such that any smooth $\mathbf{Z}_p$ action of Type II_0 on X is algebraically standard if $\overline{a}(X) = \overline{a}$ and p is a prime greater than or equal to $c(\overline{a})$.*

In contrast to Theorem 5.6. the following theorem shows a different aspect of type II_0 action.

THEOREM 5.7. [DMS]. *Let $\varepsilon_n = 1$ if $n \equiv 3(4)$ and $\varepsilon_n = 0$ otherwise. If $(n/2) - 2(\varepsilon_n + 1) \geq p$, then there are infinitely many homotopy $\mathbf{C}P^n$'s with smooth $\mathbf{Z}_p$ actions of type II_0 which are not algebraically standard. In particular, 5.2 (i) is not satisfied for such actions.*

Roughly speaking Theorem 5.7 says if the order of the group G is relatively large to compare with the dimension of the ambient manifold X, then there are non-algebraically standard type II_0 action of G.

*Sketch of proof of theorem*5.4.

Here we only sketch the proof of part (1). For part (2) and (3) we refer the reader to [D1].

CASE 1, $n = 1$:
In this case $X = S^2$ and $X^G = S^0$. Such an action is in fact G homeomorphic to a linear action. In this case Theorem 5.4 is known to be true.

CASE 2, $n = 2$:
Choose $g \in \mathbf{Z}_p$ such that the eigenvalue for the normal representation $\nu_x(F, X)$ for $x \in F$ is $\exp(i\theta)$ where $\theta = \frac{2\pi}{p}$. From Corollary 3.10 we have

$$(5.8) \qquad 1 = \mathrm{Sign}(g, X) = \pm L(F)\mathcal{L}_\theta\big(\nu(F, X)\big)[F] \pm \nu(p)$$

$$\mathcal{L}_\theta\big(\nu(F, X)\big) = a_0(1 + \sum_{j=1}^{\infty} a_j c_1(\nu)^j)$$

where $\coth(x + \frac{i\theta}{2}) = a_0(1 + \sum_{j=1}^{\infty} a_j x^j)$. An elementary calculation shows that

$$a_0 = \coth(\frac{i\theta}{2}) \qquad a_1 = \frac{-2}{i \sin\theta}$$
$$a_2 = \frac{-2}{1 - \cos\theta} \qquad a_3 = \frac{4}{i \sin\theta}\Big(\frac{\cos\theta}{1 - \cos\theta} + \frac{2}{3}\Big)$$
$$\vdots \qquad\qquad \vdots$$

By choosing appropriate integers a and b such that $1 \leq a \leq (p-1)/2$, $1 \leq b \leq (p-1)/2$, 5.8 becomes

$$\pm(1+\frac{1}{3}p_1(F)+\cdots)(a_0+a_0a_1c_1(v)+a_0a_2c_1(v)^2+\cdots)[F] + \coth\frac{ia\theta}{2}\coth\frac{ib\theta}{2}=1$$

which implies

$$\pm a_0a_1c_1(\nu)[F]+\coth\frac{ia\theta}{2}\coth\frac{ib\theta}{2}=1.$$

Thus we have

$$\pm\frac{2c_1(\nu)[F]}{1-\cos\theta}+\coth\frac{ia\theta}{2}\cdot\coth\frac{ib\theta}{2}=1. \tag{5.9}$$

At this moment we need the following number theoretical result.

LEMMA 5.10. [D1]. *Let p be an odd prime, and let $\theta = 2\pi/p$. Let a and b be as above. There is a unique integral solution for the equation*

$$\frac{2c}{1-\cos\theta}+\coth\frac{ia\theta}{2}\cdot\coth\frac{ib\theta}{2}=1$$

which is $c=-1$, $a=1$, $b=1$.

5.2 (iii) follows from $a = 1$ and $b = 1$. Lemma 5.10 also implies that $c_1(\nu)$ generates $H^2(F;\mathbf{Z})$. In the proof of Lemma 5.3 we found out that $c_1(\nu)=j^*j_!(1_F)$ and $j_!(1_F)=D_x(F)\cdot x$. Thus $\pm1=c_1(\nu)[F]=D_x(F)\cdot(j^*x)[F]$. Since $(j^*x)[F]$ is an integer $D_x(F)=1$ with an appropriate choice of orientations. This is 5.2 (ii). 5.2 (i) is trivial.

CASE 3, $n=3$:
By Hirzebruch's signature formula $p_1(F)[F]=3$. Choose $g\in\mathbf{Z}_p$ as in Case 2.

From Corollary 3.10 we have

$$\pm\cot(\frac{\theta}{2})(1-\frac{2}{1-\cos\theta}c_1^2[F])=\pm\cot(\frac{a\theta}{2})\cot(\frac{b\theta}{2})\cot(\frac{c\theta}{2})$$

Here $c_1 = c_1(\nu(F,X))$ and a,b,c are integers between 1 and $(p-1)/2$. As in case 2 we need the following lemma.

LEMMA 5.11. [D1]. *Let p,θ,a,b,c be as above. The only integral solution for the equation*

$$\left|\cot\left(\frac{\theta}{2}\right)\left(1-\frac{2k}{1-\cos\theta}\right)\right| = \left|\cot\left(\frac{a\theta}{2}\right)\cot\left(\frac{b\theta}{2}\right)\cot\left(\frac{c\theta}{2}\right)\right|$$

is $k=+1$ and $a,b,c,\in\{1,p-1\}$.

Thus 5.2 (iii) is satisfied. Since $1 = c_1^2(\nu)[F] = D_x(F)^2(j^*x)[F]$ we have $D_x(F)=1$ with an appropriate choice of orientations. Since $D_x(F)=1$ $p_1(F)[F]=3$ implies

$$p_1(F) \equiv 3j^*(x^2) \quad \text{(mod torsion)}$$

we have

$$p(F) \equiv (1+j^*x^2)^3 \quad \text{(mod torsion)}$$

Sketch of the proof of theorem 5.6.

We may assume that the eigenvalue of $z = \exp(\frac{2\pi i}{p}) \in G$ for the normal representation $\nu_x(F,X)$ for $x\in F$ is z. Thus from Corollary 3.10 we have

$$\mathrm{Sign}(z,X) = L(F)\mathcal{L}_\theta(\nu(F,X))[F] + \prod_{j=1}^{n}\frac{z^{m_j}+1}{z^{m_j}-1}$$

where $1\le |m_j| \le \frac{p-1}{2}$. Here

$$\begin{aligned}\mathcal{L}_\theta(\nu(F,X)) &= \coth\left(c_1(\nu)+\frac{\pi i}{p}\right)\\ &= \frac{z^{-1}e^{2c_1(\nu)}+1}{z^{-1}e^{2c_1(\nu)}-1} = 1-\frac{2z}{z-1}\sum_{k=0}^{n-1}\left(\frac{e^{2c_1(\nu)}-1}{z-1}\right)^k\end{aligned}$$

Since $L(F)[F] = \operatorname{Sign} F$, the above equation becomes

$$(5.12)0 = (z-1)^n(\operatorname{Sign}(z,x) - \operatorname{Sign} F) + 2z\Big\{\sum_{k=0}^{n-1}(z-1)^{n-1-k} \times \left(e^{2c_1(\nu)}-1\right)^k\Big\}L(F)[F] - (z-1)^n\prod_{j=1}^{n}\left(z^{m_j}+1\right)\Big/\left(z^{m_j}-1\right).$$

It is easy to see from the definition of signature that $|\operatorname{Sign}(z,x) - \operatorname{Sign} F| \leq s_X$ where s_X is the sum of all Betti numbers of X.

Now assume that p is sufficiently large. Then z is quite close to 1. Thus the first term of (5.12) is approximately equal to 0. An extensive estimation shows that the second term of (5.12) is approximately equal to $2^n\left(D_x(F)\right)^n$. Consider the third term.

Since $\left|(z^m-1)/(z^m+1)\right| = \left|\tan m\pi/p\right|$

$$\begin{aligned}\text{the third term } &= \left|(z+1)^n(z-1)^n/(z+1)^n\prod(z^{m_j}+1)/(z^{m_j}-1)\right| \\ &< 2^n\prod\left|\tan(\pi/p)/\tan(m_j\pi/p)\right|.\end{aligned}$$

Since $1 \leq |m_j| \leq \frac{p-1}{2}$, unless $|m_j| = 1$ for all j,

$$\begin{aligned}\text{the third term } &< 2^n\left|\tan\pi/p/\tan 2\pi/p\right| \\ &= 2^{n-1}\left(1+\tan^2\pi/p\right).\end{aligned}$$

Thus unless $|m_j| = 1$, $2^n\left(D_x(F)\right)^n$ is approximately equal to $2^{n-1}(1 + \tan^2\pi/p) < 2^n$ for sufficiently large p which is impossible. Thus the equation 5.12 hold for large p if $|m_j| = 1$ and $D_x(F) = 1$. Again by an appropriate choice of orientations we may assume that $m_j = 1$ for all j. This proves 5.2. (ii) and (iii).

Since the $\mathbf{Z}_p$ action on X induces the trivial $\mathbf{Z}_p$ action on $\mathrm{H}^n(X\,;\,\mathbf{Q})$ we have $\operatorname{Sign}(z,X) = \operatorname{Sign} X$. Thus (5.12) becomes

$$(5.13)0 = (z-1)^n(\operatorname{Sign} X - \operatorname{Sign} F) + 2z\Big\{\sum_{k=0}^{n-1}(z-1)^{n-1-k} \times (e^{2x_0}-1)^k\Big\}L(F)[F] - (z+1)^n.$$

Here $x_0 \equiv j^*x$ (mod torsion). This is an equation of z whose degree is at most n. On the other hand $z^p = 1$. Thus the coefficients of z^j in (5.13) must identically vanish if $p \geq n+2$. Therefore

$$\text{the constant term } = (\operatorname{Sign} X - \operatorname{Sign} F) - (-1)^n = 0.$$

Hence $\operatorname{Sign} X - \operatorname{Sign} F = (-1)^n$. Putting this into (5.13), we have

$$2z\Big\{\sum_{k=0}^{n-1}(z-1)^{n-1-k}(e^{2x_0}-1)^k\Big\}L(F)[F] = (1+z)^n - (1-z)^n.$$

Compare the coefficients of z^j inductively on j. The value of $(e^{2x_0} - 1)^k L(F)[F]$ are uniquely determined for each k. This determines $L(F)$ uniquely because $L(F)$ is a polynomial of x_0. On the other hand type II_0 linear actions have $p(F) = (1+x_0^2)^n$. Thus 5.2 (i) is true in general.

Sketch of the proof of theorem 5.7.

Consider the type II_0 linear action of $\mathbf{Z}_p$ on $\mathbf{C}P^n$ with the fixed point set $\mathbf{C}P^{n-1} \amalg P$. Let ν be the normal bundle of $\mathbf{C}P^{n-1}$ in $\mathbf{C}P^n$. Then the sphere bundle $S(\nu)$ is equal to S^{2n-1}. Moreover $\mathbf{Z}_p$ acts linearly and freely on $S(\nu)$. We can radially extend this linear action on D^{2n}. Actually $\mathbf{C}P^n$ is G diffeomorphic to $D(\nu) \cup_{S(\nu)} D^{2n}$. Let $f: F \longrightarrow \mathbf{C}P^{n-1}$ be any homotopy equivalence. Then $s(f^*\nu)$ is a homotopy sphere with free $\mathbf{Z}_p$ action. If $s(f^*\nu)$ is G diffeomorphic to the above $S(\nu) = S^{2n-1}$, since we can extend the action of $\mathbf{Z}_p$ on $S(f^*\nu)$ to D^{2n}, $X = D(f^*\nu) \cup_{S(f^*\nu)} D^{2n}$ is a homotopy $\mathbf{C}P^n$ with type II_0 action of $\mathbf{Z}_p$ such that $X^G = F \amalg P$. Thus if we start with F such that $p(F) \neq p(\mathbf{C}P^{n-1})$, the above $\mathbf{Z}_p$ action on X is not algebraically standard.

It remains to show when $S(f^*\nu)$ is G diffeomorphic to $S(\nu) = S^{2n-1}$. Since $\mathbf{Z}_p$ acts freely and linearly on $S(\nu) = S^{2n-1}$ the orbit space $L = S(\nu)/\mathbf{Z}_p$ is a standard lens space.

Since $\mathbf{Z}_p$ also acts freely on $S(f^*\nu)$ the orbit space $L' = S(f^*\nu)/\mathbf{Z}_p$ is a fake lens space. This implies that a homotopy equivalence $f: F \longrightarrow \mathbf{C}P^{n-1}$ induces a homotopy equivalence $h: L' \longrightarrow L$. Let N be a fixed oriented manifold. Take all possible (M, f)'s such that M is an oriented manifold and $f: M \longrightarrow N$ is a homotopy equivalence of degree

1. Introduce an equivalence relation by saying that $(M,f) \sim (M',f')$ if there is a diffeomorphism $\varphi : M \longrightarrow M'$ such that f is homotopic to $f' \circ \varphi$. Let $hS(N)$ be the set of equivalence classes of (M,f)'s.

From the previous observation we have the map

$$\pi : hS(\mathbf{C}P^{n-1}) \longrightarrow hS(L)$$

defined by $\pi\{(F,f)\} = \{(L',h)\}$. In general the set $hS(N)$ does not have a group structure. But the set $hS(N)$ has a unique zero represented by the identity map $Id : N \to N$. Let Ker $\pi = \pi^{-1}\Big(\{(L,Id)\}\Big)$. Classical surgery theoretic argument shows the following lemma.

LEMMA 5.14. [DMS]. *Under the same hypothesis as theorem 5.7*

$$|Ker\ \pi| = \infty$$

Thus there are infinitely many homotopy equivalences $f : F \longrightarrow \mathbf{C}P^{n-1}$ such that $S(f^*\nu)/\mathbf{Z}_p$ is diffeomorphic to a standard lens space L. In this case the action of $\mathbf{Z}_p$ on $S(f^*\nu)$ is free linear, which is G diffeomorphic to a linear action on $S(\nu) = S^{2n-1}$. A classical result of Wall [Wa] says that diffeomorphism types of homotopy $\mathbf{C}P^m$'s are distinguished by their Pontrjagin classes up to finite ambiguity. Thus there are infinitely many homotopy equivalences $f : F \longrightarrow \mathbf{C}P^{n-1}$ with $p(F) \neq p(\mathbf{C}P^{n-1})$ such that F appears as a component of fixed point set of type II_0 action on homotopy $\mathbf{C}P^n$ X.

REMARK 5.15. *Actually Lemma 5.14 is proved for a modified version of $hS(N)$. Namely in the definition of $hS(N)$ the map $f : M \longrightarrow N$ is a simple homotopy equivalence. So, to be precise, all homotopy equivalence in the proof of Theorem 5.7 should be simple homotopy equivalences. But the above way does not cause a serious difference in the general idea of the proof.*

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