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DERIVATIONS AND EPIMORPHISMS ON BANACH ALGEBRAS

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1. Introduction

A derivation on a Banach algebra A is a linear mapping D of A into itself such that $Dab = a(Db)+(Da)b$, $(a, b \in A)$. If T is a linear operator from a Banach space X into a Banach space Y, then the separating space $\mathcal{S}(T)$ of T is defined by $\mathcal{S}(T) = \{y \in Y$: there is a sequence $\{x_n\}$ in X such that $x_n \to 0$ in X with $T_{x_n} \to y$ in $Y\}$. It is well known that $\mathcal{S}(T)$ is a ideal of Y if $T : X \to Y$ has dense range, and $T : X \to Y$ is continuous if and only if $\mathcal{S}(T) = \{0\}$.

In [8] Singer and Wermer showed that the range of a continuous derivation on a commutative Banach algebra is contained in the radical and conjectured that the assumption of continuity is unnecessary. In [3] Cusack showed that if any derivation D on a commutative Banach algebra A has a nilpotent separating space, then the range of D is contained in the radical of A. By the radical of a commutative Banach algebra A, or $rad(A)$, we mean the intersection of the kernels of all irreducible representations of A or equivalently the intersection of all maximal modular ideals of A. Hence if A has a unit, then the radical of A is the intersection of all maximal ideals.

A closed ideal I of a Banach algebra A is called a separating ideal if, for every sequence (a_n) in A, there is a positive integer N such that

$$(Ia_1a_2\ldots a_n)^- = (Ia_1a_2\ldots a_N)^- \quad \text{for} \quad n \geq N,$$

where $(Ia_1a_2\ldots a_n)^-$ is the closure of $(Ia_1a_2\ldots a_n)$.

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In this paper we show that if A is a commutative Banach algebra and if D is a derivation on A, then the range of D on $rad(A)$ is contained in $rad(A)$ if and only if the range of D on A is contained in the redical. Also we show that if K is defined by

$$K = \{x \in rad(A) : \text{ for every } n \geq 1,\ D^n x \in rad(A)\},$$

then the range of D on $\overline{K}$ is contained in $\overline{K}$ if and only if range of D on A is contained in the radical of A, which generalizes a main result of [9], also it is a characterization of [5]. In §3, we discuss the continuity of a homomorphism on Banach algebras. We prove that if A is a Banach algebra and B is a semi-prime Banach algebra and Q is a homomorphism from A onto B such that $\cap_{n\geq 1}(\mathcal{S}(Q)^n)^- = \{0\}$, then Q is continuous.

Throughout this paper, we suppose that A is a commutative Banach algebra and let D be a derivation on A, R and L will denoted, respectively, the radical and the prime radical of A.

2. Separating spaces of derivations

LEMMA 2.1.[3,4,7]. *If the radical of the separating spaces $\mathcal{S}(D)$ of D is nilpotent, then $\mathcal{S}(D)$ is nilpotent. In particular, the range of D on A is contained in R.*

THEOREM 2.2. *If $\cap_{n\geq 1}((\mathcal{S}(D) \cap R)^n)^- = \{0\}$, then $DA \subseteq R$.*

Proof. Let $x \in \mathcal{S}(D) \cap R$. Since $\mathcal{S}(D)$ is a separating ideal of A, there is a positive integer N such that

$$(\mathcal{S}(D)x^N)^- = (\mathcal{S}(D)x^n)^- \quad \text{for} \quad n \geq N.$$

Since $\mathcal{S}(D)x^{n+1} \subseteq \mathcal{S}(D)x^n$ for $n \geq 1$, then

$$(\mathcal{S}(D)x^N)^- = \cap_{n\geq 1}(\mathcal{S}(D)x^n)^- \subseteq \bigcap_{n\geq 1}((\mathcal{S}(D) \cap R)^n)^- = \{0\}.$$

Therefore $x^{N+1} = 0$. Thus $\mathcal{S}(D) \cap R$ is a closed nil ideal of A and so $\mathcal{S}(D) \cap R$ is nilpotent. Hence $\mathcal{S}(D)$ is nilpotent and we have $DA \subseteq R$.

COROLLAY 2.3. *If* $\cap_{n\geq 1}(\mathcal{S}(D)^n)^- = \{0\}$, *then* $DA \subset R$.

COROLLARY 2.4. *If* $\cap_{n\geq 1}((\mathcal{S}(D)\cap R)^n)^- = \{0\}$ *and* R *is an integral domain, then* D *is continuous.*

Proof. By theorem 2.2, $\mathcal{S}(D)$ is nilpotent. Therefore $\mathcal{S}(D)$ is contained in the prime radical L of A. Since $L \subseteq R$ and R is an integral doamin, $L = \{0\}$. Thus $\mathcal{S}(D) = \{0\}$ and so D is continuous.

LEMMA 2.5.[8]. *Let* S *be a linear mapping from a Banach space* X *into a Banach space* Y *and let* X_0 *and* Y_0 *be closed linear subspaces of* X *and* Y, *respectively, such that* SX_0 *is contained in* Y_0. *If* $S_0 : X/X_0 \to Y/Y_0$ *is defined by*

$$S_0(x + X_0) = S_x + Y_0.$$

Then S_0 *is continuous if and only if* $\mathcal{S}(D)$ *is contained in* Y_0.

THEOREM 2.6. *If* $D(\mathcal{S}(D)) \subseteq \mathcal{S}(D)$ *and* $\mathcal{S}(D) \subseteq R$, *then* $DA \subseteq R$.

Proof. If $D(\mathcal{S}(D)) \subseteq \mathcal{S}(D)$ then D gives a derivation D_1 on $A/\mathcal{S}(D)$ defined by

$$D_1(a + \mathcal{S}(D)) = D_a + \mathcal{S}(D) \quad (a \in A).$$

Since $\mathcal{S}(D) \subseteq \mathcal{S}(D)$, by Lemma 2.5, D_1 is continuous. Therefore $D_1(A/\mathcal{S}(D)) \subseteq rad(A/\mathcal{S}(D))$ and $D_1(A/\mathcal{S}(D)) \subseteq R/\mathcal{S}(D)$, which completes the proof.

COROLLARY 2.7. *If* $\mathcal{S}(D)^2 = \mathcal{S}(D)$ *and* $\mathcal{S}(D) \subseteq R$, *then* $DA \subseteq R$.

Proof. Let $x = ab$ for $a, b \in \mathcal{S}(D)$. Then $Dab = a(Db)+(Da)b \in \mathcal{S}(D)$, because $\mathcal{S}(D)$ is an ideal of A. By Lemma 2.5, and Theorem 2.6, we complete the proof.

LEMMA 2.8.[5]. *If* K *is the ideal defined by* $K = \{x \in R : D^n x \in R$ *for all* $n \geq 1\}$, *then* K *is a prime ideal of* A.

LEMMA 2.9. *If* D *is a derivation on* A, *then* $D^n R^{n+1} \subseteq R$.

Proof. Let $x \in R^{n+1}$. Then

$$x = \sum_{i=1}^{k} a_{i_1} \dots a_{i_{n+1}} \quad \text{for} \quad \{a_{ij}\}_{i=1,\dots,k}^{j=1,\dots,n+1} \quad \text{in} \quad R.$$

We prove that $D^n a_1 \dots a_{n+1} \in R$ for $a_1, \dots, a_{n+1} \in R$. For $n = 1$, $Da_1a_2 = a_1(Da_2) + (Da_1)a_2 \in R$, $(a_1, a_2 \in R)$. Suppose that $D^k a_1 \dots a_{k+1} \in R$, $(a_1, \dots, a_{k+1} \in R)$. Then $D^{k+1}a_1 \dots a_{k+2} = D^k((a_1 \dots a_{k+1})Da_{k+2} + a_{k+2}D(a_1 \dots a_{k+1})) \in R$, $(a_1, \dots, a_{k+2} \in R)$. Therefore $D^n a_1 \dots a_{n+1} \in R$, $(a_1, \dots, a_{n+1} \in R)$. Since D^n is linear, $D^n x \in R$.

LEMMA 2.10.[5]. *If* $\cap_{n \geq 1} R^n = \{0\}$, *then* $DA \subseteq R$.

THEOREM 2.11. $DA \subseteq R$ *if and onky if* $D\overline{K} \subseteq \overline{K}$.

Proof. If $DA \subseteq R$, then $K = R$. Therefore $D\overline{K} = \overline{K}$. Conversely if $D\overline{K} \subseteq \overline{K}$, then D gives a derivation D_1 on $A/\overline{K}$ defined by $D_1(a+\overline{K}) = Da + \overline{K}$, $(a \in A)$. Since $\overline{K} \subseteq R$, $rad(A/\overline{K}) = R/\overline{K}$. Now we prove that $\cap_{n \geq 1}(R/\overline{K})^n = \{0\}$. Let $x \in \cap_{n \geq 1}(R/\overline{K})^n$. Then there are $a_n \in R^n$ such that $x = a_n + \overline{K}$. Now for every $n \geq 1$, $a_1 - a_{n+1} \in \overline{K}$. Hence $D^n(a_1 - a_{n+1}) \in \overline{K}$ and by Lemma 2.9, $D^n a_{n+1} \in R$. Thus $D^n a_1 \in R$ for every $n \geq 1$. Therefore $a_1 \in K \subseteq \overline{K}$ and so $x = 0$ in $A/\overline{K}$. Thus $DA \subseteq R$.

Clearly, we have the following corollary.

COROLLARY 2.12. $DA \subseteq R$ *if and only if* $DR \subseteq R$.

From Theorem 2.11, Corollary 2.12 and [5], we get the following result.

COROLLARY 2.13. *The following statements are equivalent:*
(a) $DA \subseteq R$
(b) $S(D) \cap R \subseteq K$
(c) K *is closed*
(d) $D\overline{K} \subseteq \overline{K}$
(e) $DR \subseteq R$.

From this Corollary 2.13, we get the following. This result is a main theorem of [9].

COROLLARY 2.14.[9]. *Every derivation on a commutative Banach algebra* A *in which every prime ideal is closed has a range in its radical.*

3. Separating spaces of homomorphisms

Let A and B be Banach algebras. By an epimorphism Q from A onto B we mean a linear mapping satisfying $Q(ab) = Q(a)Q(b)$, $(a, b \in A)$.

THEOREM 3.1. *If* $\cap_{n\geq 1}(\mathcal{S}(Q)^n)^- = \{0\}$, *then the separating space* $\mathcal{S}(Q)$ *of* Q *is nilpotent.*

Proof. Let $x \in \mathcal{S}(Q)$, since $\mathcal{S}(Q)$ is a separating ideal of B, there is a positive integer N such that

$$(\mathcal{S}(Q)x^N)^- = (\mathcal{S}(Q)x^n)^- \quad (n \geq N).$$

Since $\mathcal{S}(Q)x^{n+1} \subseteq \mathcal{S}(Q)x^n$, $(n \geq 1)$, $(\mathcal{S}(Q)x^N)^- = \cap_{n\geq 1}(\mathcal{S}(Q)x^n)^- \subseteq \cap_{n\geq 1}(\mathcal{S}(Q)^n)^- = \{0\}$. Therefore $x^{N+1} = 0$ and so $\mathcal{S}(Q)$ is nil. Since $\mathcal{S}(Q)$ is a closed nil ideal of B, $\mathcal{S}(Q)$ is nilpotent.

COROLLARY 3.2. *If* $\cap_{n\geq 1}(\mathcal{S}(Q)^n)^- = \{0\}$ *and if* R *is an integral domain, then* Q *is continuous.*

Proof. Compare with Corollary 2.4.

COROLLARY 3.3. *If* $\cap_{n\geq 1}(\mathcal{S}(Q)^n)^- = \{0\}$ *and* B *is semi-prime, then* Q *is continuous.*

Proof. By Theorem 3.1, $\mathcal{S}(Q)$ is nilpotent. Therefore $\mathcal{S}(Q)$ is contained in the prime radical L of B. Since B is semi-prime, $L = \{0\}$. Hence $\mathcal{S}(Q) = \{0\}$ and so Q is continuous.

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