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A CURVILINEAR EXTENSION OF THE MAXIMUM PRINCIPLE FOR HARMONIC FUNCTIONS

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Let D be a simply connected domain in the z-plane, which is not the whole plane, Consider non-to-one conformal mapping $z = f(w)$ of the unit disc $|w| < 1$ onto D. Caratheodory showed that it induces a one-to-one correspondence between the points of $|w| = 1$ and the prime ends of D such that if P is a prime end of D and γ is a curve at P, then $f^{-1}(\gamma)$ is an arc in D at the point P_w that corresponds to the prime end P.

A set $\tilde{E}$ of prime ends of D is called a D-conformal null set if it corresponds to a set of linear measure zero on $|w| = 1$. Since the set of all prime ends of D which are not determined by any accessible point of D is a D-conformal null set, a set $\tilde{E}$ of prime ends of D is a D-conformal null set if and only if the prime ends in $\tilde{E}$ which are determined by accessible points is a D-conformal null set.

A set of accessible points of a simply connected domain is a D-conformal null set if it corresponds to a set of linear measure zero on the unit circle under a one-to-one conformal mapping.

In [1] we have shown the following

THEOREM 1. *Let D be a simply connected domain in the z-plane, which is not the whole plane, ζ_0 a boundary point of D, $\tilde{E}$ a conformal null set of prime ends of D. If $f(z)$ is meromorphic in D and bounded in the intersection of D with some neighborhood of ζ_0, then*

$$\limsup_{\substack{z\to\zeta_0 \\ z\in D}} |f(z)| = \limsup_{\substack{z(a)\to\zeta_0 \\ P(a)\in\tilde{D}-\tilde{E}}} \left(\inf_{\Lambda}\left(\limsup_{\substack{z\to z(a) \\ z\in\Lambda}} |f(z)|\right)\right)$$

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where $\wedge$ is an arc at an accessible boundary point a with $P(a) \in \tilde{D} - \tilde{E}$ and the convergence is the sense of the ordinary euclidean metric.

The following theorem is an immediate corollary of Theorem 1.

THEOREM 2. *Let D be a simply connected domain in the z-plane, which is not the whole plane, and ζ_0 a boundary point of D. If $u(z)$ is harmonic in D and bounded above in the intersection of D with some neighborhood of ζ_0, then*

$$\limsup_{\substack{z \to \zeta_0 \\ z \in D}} u(z) = \lim_{\substack{z(a) \to \zeta_0 \\ P(a) \in \tilde{D} - \tilde{E}}} \sup(\inf_{\wedge}(\limsup_{\substack{z \to z(a) \\ z \in \wedge}} u(z))$$

where $\wedge$ is an arc at an accessible boundary point a with $P(a) \in \tilde{D} - \tilde{E}$ and the convergence is in the sense of the ordinary euclidean metric.

Proof. Let $v(z)$ be the harmonic conjugate of $u(z)$. Then the function $f(z) = \exp(u + iv)$ is an analytic function in D and bounded in the intersection of D with some neighborhood of ζ_0.

Applying Theorem 1 to $f(z)$, we obtain

$$\limsup_{\substack{z \to \zeta_0 \\ z \in D}} |f(z)| \leq e^m,$$

that is,

$$\limsup_{\substack{z \to \zeta_0 \\ z \in D}} \exp(u(z)) \leq e^m$$

Hence we obtain

$$\lim_{\substack{z \to \zeta_0 \\ z \in D}} u(z) \leq m.$$

We state several corollaries of Theorem 1 and Theorem 2.

COROLLARY 1. *Let D be a simply connected domain in the z-plane, which is not the whole plane, ζ_0 a boundary point of D, $\tilde{E}$ a conformal null set of prime ends of D. If $u(z)$ is harmonic in D and bounded above in the intersection of D with some neighborhood of ζ_0, $N(\zeta_0)$, and at* *each accessible boundary point a with $P(a) \in \tilde{D} - \tilde{E}$, $z(a) \in \partial D \cap N(\zeta_0)$, there exists an arc Λ_a at $z(a)$ determining a on which*

$$\lim_{\substack{z \to z(a) \\ z \in \Lambda_a}} \sup u(z) \leq m$$

then

$$\lim_{\substack{z \to z(a) \\ z \in D}} \sup u(z) \leq m$$

COROLLARY 2. *Let D be a simply connected domain in the z-plane, which is not the whole plane, and let ζ_0 be a boundary point of D, E a subset of D such that the set $\{p(a) : z(a) \in E$, a is an accessible boundary point of $D\}$ is a D-conformal null set. If $u(z)$ is harmonic in D and bounded above in the intersection of D with some neighborhood of ζ_0, then*

$$\lim_{\substack{z \to \zeta_0 \\ z \in D}} \sup u(z) = \lim_{\substack{z(a) \to \zeta_0 \\ Z(a) \to \partial D - E}} \sup(\inf_{\substack{\Lambda_a \\ \Lambda_a \to \zeta}} (\lim_{\substack{z \to z(a) \\ z \in \Lambda_a}} \sup u(z))),$$

where Λ_a is an arc at an accessible boundary point a with $z(a) \in \partial D - E$ and the convergence is in the sense of the ordinary euclidean metric.

COROLLARY 3. *If E is of $\frac{1}{2}$-dimensional Hausdorff measure zero in the boundary of D, in place of the assumption on E in Corollary 2, then the same conclusion holds as in Corollary 2.*

COROLLARY 4. *If E is of logarithmic capacity zero, in place of the assumption on E in Corollary 2, then the same conclusion holds as Corollary 2.*

COROLLARY 5. *Under the same hypothesis as in Corollary 2 if the set E of exceptional points is of capacity zero, the same conclusion holds as in Corollary 2.*

References

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