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ON JOINT SPECTRA OF ELEMENTS IN A C^*-ALGEBRA

YOUNGOH YANG

1. Introduction

In [5], Harte introduced the joint spectrum of an n–tuple of elements in a unital Banach algebra and studied the spectral mapping theorem. In [3], Fujii and Lin introduced the normalized topological divisors of zero and some characterizations of normal approximate spectra. Motivated by results in [5], [6], we shall introduce several joint spectra of an n-tuple $a = (a_1, \ldots, a_n)$ of elements in a unital C^*-algebra A, and investigate the relations among these spectra, and the numerical range in this note.

Throughout this paper, all algebras will be over the complex field $\mathbf{C}$. Algebras are assumed to have an identity e.

2. Relations among joint spectra

We will make use of the notion of 'joint spectrum' (but omit the word 'joint') as it was introduced for example by Harte [5].

DEFINITION 2.1. *Let A denote a unital normed algebra. If $a =$*

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$(a_1,\dots,a_n)\in A^n$ *is an* n*-tuple of elements of* A*, we call*

$$\sigma_l(a)=\{\lambda\in\mathbf{C}^n : e\notin\sum_{j=1}^n A(a_j-\lambda_j)\}\ \textit{the left spectrum of } a,$$

$$\sigma_\gamma(a)=\{\lambda\in\mathbf{C}^n : e\notin\sum_{j=1}^n (a_j-\lambda_j)A\}\ \textit{the right spectrum of } a,$$

$$\sigma(a)=\sigma_l(a)\cup\sigma_\gamma(a)\ \textit{the spectrum of } a,$$

$$L(a)=\{\lambda\in\mathbf{C}^n : \inf_{\substack{x\in A\\ \|x\|=1}}\sum_{j=1}^n\|(a_j-\lambda_j)x\|=0\}$$

the left approximate point spectrum of a*,*

$$R(a)=\{\lambda\in\mathbf{C}^n : \inf_{\substack{x\in A\\ \|x\|=1}}\sum_{j=1}^n\|x(a_j-\lambda_j)\|=0\}$$

the right approximate point spectrum of a*.*

These spectra are natural generalizations from the case $n=1$, and so inherit many of the properties of the spectra of a single element as we now recall from [3], [5], [7].

For $a=(a_1,\dots,a_n)\in A^n$, we say that λ belongs to $LR(a)$ if there exists a sequence $\{y_k\}$ of unit elements in A such that $(a_j-\lambda_j)y_k\to 0$ and $y_k(a_j-\lambda_j)\to 0$ as $k\to\infty$ $(j=1,\dots,n)$.

The following lemma follows from the definitions;

LEMMA 2.2. *For a fixed* n*-tuple* $a=(a_1,\dots,a_n)$ *of elements in a unital normed algebra* A*, let* f_a *and* g_a *be two functions on* $\mathbf{C}^n$ *defined by*

$$f_a(\lambda)=\inf_{x\in A}\Big(\sum_{j=1}^n\|(a_j-\lambda_j)x\|/\|x\|\Big)$$

and

$$g_a(\lambda)=\inf_{x\in A}\Big(\sum_{j=1}^n\|x(a_j-\lambda_j)\|/\|x\|\Big).$$

Then

(1) $\lambda \in L(a)$ *iff* $f_a(\lambda) = 0$, *and* $\lambda \in R(a)$ *iff* $g_a(\lambda) = 0$.

(2) f_a *and* g_a *are continuous. In fact we have*

$$|f_a(\lambda) - f_a(\mu)| \leq \sqrt{n}|\lambda - \mu| \text{ and } |g_a(\lambda) - g_a(\mu)| \leq \sqrt{n}|\lambda - \mu|.$$

In what follows, unless exception is noted, A denotes an abstract C^*-algebra with identity e. If there is no ambiguity, we shall write $\sum$ for $\sum_{j=1}^{n}$.

DEFINITION 2.3. $a = (a_1, \ldots, a_n) \in A^n$ *is called a jointly normalized topological divisor of zero, briefly, JNTDZ, if there exists a sequence* $\{x_k\}$ *of unit elements in* A *such that* $a_j x_k \to 0$ *and* $a_j^* x_k \to 0$ *as* $k \to \infty$ *for* $j = 1, \ldots, n$. *For a fixed* $a \in A^n$, *we define* $N(a) = \{\lambda \in \mathbf{C}^n : a - \lambda$ *is a JNTDZ*$\}$.

It is obvious that $LR(a) \subseteq L(a) \cap R(a)$.

THEOREM 2.4. *For* $a = (a_1, \ldots, a_n) \in A^n$, $N(a) \subseteq L(a) \cap R(a) \subseteq L(a) \cup R(a) = \sigma(a)$.

Proof. This follows from the definitions and Lemma 2.3 [7].

It is known that $\sigma_l(a) = L(a)$, $\sigma_\gamma(a) = R(a)$, and $\sigma(a)$ are nonempty if $a = (a_1, \ldots, a_n) \in A^n$ is a commuting n-tuple [5], [7].

COROLLARY 2.5. *For* $a = (a_1, \ldots, a_n) \in A^n$ *and* $\lambda \in \mathbf{C}^n$,

(1) $\lambda \in \sigma(a)$ *iff* $f_a(\lambda) = 0$ *or* $g_a(\lambda) = 0$, *where* f_a *and* g_a *are as in Lemma 2.2.*

(2) $\lambda \in N(a)$ *iff* $p_a(\lambda) = 0$, *where*

$$p_a(\lambda) = \inf_{x \in A}\{\sum(\|(a_j - \lambda_j)x\| + \|(a_j - \lambda_j)^* x\|)/\|x\|\}.$$

(3) $\lambda \in LR(a)$ *iff* $q_a(\lambda) = 0$, *where*

$$q_a(\lambda) = \inf_{x \in A}\{\sum(\|(a_j - \lambda_j)x\| + \|x(a_j - \lambda_j)\|)/\|x\|\}.$$

(4) If a is a commuting n-tuple, then $N(a)$, $L(a)$, $R(a)$ and $LR(a)$ are all compact subsets of $\sigma(a)$.

Proof. (1), (2) and (3) are obvious.
(4) To show that $L(a)$ is closed in $\sigma(a)$, we observe that if $\lambda \notin L(a)$, then any $\mu \in \mathbf{C}^n$ with $\sqrt{n}|\lambda - \mu| < f_a(\lambda)$ is not in $L(a)$ because $0 < f_a(\lambda) - \sqrt{n}|\lambda - \mu| \leq f_a(\mu)$ by the above Lemma. This shows that $L(a)$ is in $\sigma(a)$ and hence compact. Similarly, $R(a)$ is compact. As for $N(a)$ and $LR(a)$, we have $p_a(\lambda) \leq n\sqrt{n}|\mu - \lambda| + p_a(\mu)$ i.e., $|p_a(\lambda) - p_a(\mu)| \leq n\sqrt{n}|\lambda - \mu|$ and similarly $|q_a(\lambda) - q_a(\mu)| \leq n\sqrt{n}|\lambda - \mu|$. The same argument as above shows that $N(a)$ and $LR(a)$ are compact.

COROLLARY 2.6. *For a fixed $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$, the following sets are closed in A^n.*

$$\{a \in A^n : \inf_{\substack{x \in A \\ \|x\|=1}} \sum \|(a_j - \lambda_j)x\| = 0\},$$

$$\{a \in A^n : \inf_{\substack{x \in A \\ \|x\|=1}} \sum \|x(a_j - \lambda_j)\| = 0\},$$

$\{a \in A^n : a - \lambda$ is a JNTDZ$\}$, and the set of all $a \in A^n$ such that there exists a sequence $\{y_k\}$ of unit elements in A satisfying $(a_j - \lambda_j)y_k \to 0$ and $y_k(a_j - \lambda_j) \to 0$ for $j = 1, \ldots, n$.

THEOREM 2.7. *(1) If $a = (a_1, \ldots, a_n) \in A^n$ is an n–tuple of hyponormal elements, then $N(a) = L(a) \subseteq R(a) = \sigma(a)$.*
(2) If $a = (a_1, \ldots, a_n)$ is an n–tuple of normal elements, then $\sigma(a) = L(a) = R(a) = N(a)$.

Proof. (1) $a - \lambda$ is hyponormal iff a is hyponormal. Thus in order to show that $N(a) = L(a)$, it suffices to show that if $0 \in L(a)$, then $0 \in N(a)$. Since $a_j^* a_j \geq a_j a_j^*$ for $j = 1, \ldots, n$ and A is a C^*-algebra, $(a_j y_k)^*(a_j y_k) \geq (a_j^* y_k)^*(a_j^* y_k)$ for any bounded sequence $y_k \in A$. Then for $j = 1, \ldots, n$, $\|a_j y_k\|^2 \geq \|a_j^* y_k\|^2$ since $(a_j^* y_k)^*(a_j^* y_k)$ is positive. Thus for $j = 1, \ldots, n, a_j y_k \to 0$ imply $a_j^* y_k \to 0$. Hence $0 \in N(a)$.
(2) The proof follows from Lemma 2.3 [7] and (1).

THEOREM 2.8. *Let $a = (a_1, \ldots, a_n) \in A^n$ satisfying the relation $a_j^* b_j a_j + a_j + a_j^* \geq 0$ for some n-tuple $b = (b_1, \ldots, b_n)$ of selfadjoint elements ($j = 1, \ldots, n$). If $y_k \in A$ is any bounded sequence, then for $j = 1, \ldots, n$, the relation $a_j y_k \to 0$ implies that $a_j^* y_k \to 0$ as $k \to \infty$.*

Proof. Let us write $p_j = a_j^* b_j a_j + a_j + a_j^* \geq 0$ and let $a_j y_k \to 0$ as $k \to \infty$ ($j = 1, \ldots, n$). Then for $j = 1, \ldots, n$, $(p_j - a_j^*) y_k \to 0$ as $k \to \infty$. But then

$$\|p_j y_k\|^2 \leq \|p_j^{\frac{1}{2}}\|^2 \|p_j^{\frac{1}{2}} y_k\|^2 = \|p_j^{\frac{1}{2}}\|^2 \|y_k^* p_j y_k\|,$$

and

$$\|y_k^* p_j y_k\| \leq \|y_k^* a_j^* b_j a_j y_k\| + \|y_k^* a_j y_k\| + \|y_k^* a_j^* y_k\| \to 0 \quad (j = 1, \ldots, n).$$

This shows that $p_j y_k \to 0$ and hence $a_j^* y_k \to 0$ as $k \to \infty$.

COROLLARY 2.9. *Let $a = (a_1, \ldots, a_n) \in A^n$.*

(1) If Re $a =$ (Re $a_1, \ldots,$ Re $a_n) \geq 0$, then $a_j y_k \to 0$ iff $a_j^* y_k \to 0$ for any bounded sequence $\{y_k\}$ in A.

(2) If $\lambda \in \sigma(a)$ and Re$(a - \lambda) \geq 0$, then $\lambda \in N(a)$.

For $x \in A$ let $x \to T_x$ be a faithful $*$-representation of A on a Hilbert space H. The closed numerical range $W(x)$ of x is defined by $W(x) = \overline{W(T_x)}$ = the closure of the usual numerical range of the operator T_x, i.e., the closure of $\{(T_x f, f) : f \in H, \|f\| = 1\}$ which is compact and convex. Let us denote by $\sum$ the set of all normalized states of A, (i.e., the set of all linear functionals p on A such that $p(e) = 1$ and $p(x^* x) \geq 0$ for all x in A). It is known that $W(x) \supseteq Sp(x)$, the spectrum of $x \in A$ in general, but $W(x) = \text{conv}\, Sp(x)$ = the convex hull of $Sp(x)$ whenever x is normal ([8] Theorem 8), and $W(x) = \sum(x) = \{p(x) : p \in \sum\}$ for any $x \in A$ ([1] Theorem 3).

THEOREM 2.10. *The following statements are equivalent.*

(1) $\lambda \in L(a)$ ($\lambda \in R(a)$).

(2) *There does not exist $\varepsilon > 0$ such that*

$$\sum (a_j - \lambda_j)^* (a_j - \lambda_j) \geq \varepsilon$$

$$\left(\sum (a_j - \lambda_j)(a_j - \lambda_j)^* \geq \varepsilon\right).$$

(3)

$$0 \in W(\sum(a_j - \lambda_j)^*(a_j - \lambda_j))$$
$$(0 \in W(\sum(a_j - \lambda_j)(a_j - \lambda_j)^*)).$$

(4) *There exists* $p \in \sum$ *such that*

$$\sum p((a_j - \lambda_j)^*(a_j - \lambda_j)) = 0$$
$$(\sum p((a_j - \lambda_j)(a_j - \lambda_j)^*) = 0).$$

(5)

$$\sum(a_j - \lambda_j)^*(a_j - \lambda_j) \text{ is singular in } A$$
$$(\sum(a_j - \lambda_j)(a_j - \lambda_j)^* \text{ is singular in } A).$$

(6) *There exists a sequence* $\{x_k\}$ *of unit elements in* A *such that* $\lim_{k \to \infty}(\sum \|(a_j - \lambda_j)x_k\|) = 0$

$$(\lim_{k \to \infty} \sum \|x_k(a_j - \lambda_j)\| = 0).$$

Proof. Put $c = \sum_{j=1}^{n}(a_j - \lambda_j)^*(a_j - \lambda_j)$.

(1) $\Longrightarrow$ (2). Suppose that there exists a real number $\varepsilon > 0$ such that $c \geq \varepsilon$. Using a standard argument, it follows that there exists an element b in A for which $bc = e$ and therefore $\sum A(a_j - \lambda_j) = A$.

(2) $\Longrightarrow$ (3). By a faithful $*$-representation $x \to T_x$, there does not exist $\varepsilon > 0$ such that

$$\sum(T_{a_j} - \lambda_j)^*(T_{a_j} - \lambda_j) \geq \varepsilon.$$

Since both operators on the left sides are positive, it follows easily that

$$0 \in W(\sum(T_{a_j} - \lambda_j)^*(T_{a_j} - \lambda_j)) = W(c).$$

(3) $\Longrightarrow$ (4). As $W(x) = \sum(x)$ by a previous remark, $0 \in W(c)$ iff there exists $p \in \sum$ such that $p(c) = 0$.

(4) $\Longrightarrow$ (5). As $W(x) = \text{conv } Sp(x)$ whenever x is normal, $0 \in Sp(c)$, and hence c is singular in A.

(5) $\Longrightarrow$ (6) and (6) $\Longrightarrow$ (1). These follow from Lemma 2.3 [7].

COROLLARY 2.11. *For $a = (a_1, \dots, a_n) \in A^n$ and $\lambda \in \mathbf{C}^n$, the following statements are equivalent.*

(1) $\lambda \in \sigma(a)$.

(2) *There does not exist $\varepsilon > 0$ such that either*

$$\sum(a_j - \lambda_j)^*(a_j - \lambda_j) \geq \varepsilon \text{ or } \sum(a_j - \lambda_j)(a_j - \lambda_j)^* \geq \varepsilon.$$

(3) *Either $0 \in W(\sum(a_j - \lambda_j)^*(a_j - \lambda_j))$ or*

$$0 \in W(\sum(a_j - \lambda_j)(a_j - \lambda_j)^*).$$

(4) *There exists $p \in \sum$ such that either*

$$\sum p((a_j - \lambda_j)^*(a_j - \lambda_j)) = 0$$

or

$$\sum p((a_j - \lambda_j)(a_j - \lambda_j)^*) = 0.$$

(5) *Either $\sum(a_j - \lambda_j)^*(a_j - \lambda_j)$ is singular in A or $\sum(a_j - \lambda_j)(a_j - \lambda_j)^*$ is singular in A.*

(6) *There exists a sequence $\{x_k\}$ of unit elements in A such that either $\lim_{k \to \infty} (\sum \|(a_j - \lambda_j)x_k\|) = 0$ or*

$$\lim_{k \to \infty} (\sum \|x_k(a_j - \lambda_j)\|) = 0.$$

Let $L(H)$ denote the algebra of all bounded linear operator on H and κ denote the ideal of compact operators on H. Let π be the canonical homomorphism from $L(H)$ onto the Calkin algebra $L(H)/\kappa$. If $T =$

$(T_1, \ldots, T_n)$ is an n-tuple of operators on H, then we write $\pi(T_j) = t_j$, the coset containing T_j for each $j = 1, \ldots, n$.

COROLLARY 2.12. *For an n-tuple $T = (T_1, \ldots, T_n)$ of operators on H, the following statements are equivalent.*

(1) *$\lambda \in \sigma_e(T)$, the joint essential spectrum of T.*

(2) *There does not exist $\varepsilon > 0$ such that either*

$$\sum (t_j - \lambda_j)^*(t_j - \lambda_j) \geq \varepsilon \text{ or}$$
$$\sum (t_j - \lambda_j)(t_j - \lambda_j)^* \geq \varepsilon.$$

(3) *Either*

$$0 \in W_e(\sum (T_j - \lambda_j)^*(T_j - \lambda_j)) \text{ or } 0 \in W_e(\sum (T_j - \lambda_j)(T_j - \lambda_j)^*),$$

where $W_e(T)$ denotes the essential numerical range of T.

(4) *Either*

$$0 \in Sp_e(\sum (T_j - \lambda_j)^*(T_j - \lambda_j)) \text{ or } 0 \in Sp_e(\sum (T_j - \lambda_j)(T_j - \lambda_j)^*).$$

(5) *There exists a sequence $\{x_k\}$ or unit vectors in H with $x_k \to 0$ weakly such that either*

$$\lim_{k \to \infty} (\sum \|(T_j - \lambda_j)x_k\|) = 0 \text{ or}$$
$$\lim_{k \to \infty} (\sum \|(T_j - \lambda_j)^* x_k\|) = 0.$$

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Department of Mathematics
Cheju National University
Cheju, 690–756, Korea