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ISOMORPHISM CLASSES OF REGULAR $2p$-FOLD COVERING GRAPHS

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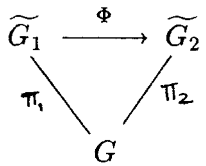
1. Introduction

Let G be a finite connected simple graph. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. We denote the set of vertices adjacent to $v \in V(G)$ by $N(v)$ and call it the neighborhood of a vertex v. The number $\beta(G) = |E(G)| - |V(G)| + 1$ is equal to the number of independent cycles in G and it is referred to the Betti number of G.

A graph $\widetilde{G}$ is called a covering of G with the projection $\Pi : \widetilde{G} \to G$ if there is a surjection $\Pi : N(\widetilde{v}) \to N(v)$ which is a bijection for any vertex $v \in V(G)$ and $\widetilde{v} \in \Pi^{-1}(v)$. We say that $\widetilde{G}$ is an n-fold covering of G if the projection $\Pi|_{V(\widetilde{G})}$ is n-to-one. Two coverings $\Pi_i : \widetilde{G_i} \to G$, $i = 1, 2$, are called isomorphic if there exists a graph isomorphism $\Phi : \widetilde{G_1} \to \widetilde{G_2}$ such that the diagram commutes.



we denote it by $\widetilde{G_1} \cong \widetilde{G_2}$.

A covering $\Pi : \widetilde{G} \to G$ is said to be regular if there is a subgroup A of the automorphism group $\mathrm{Aut}(\widetilde{G})$ of $\widetilde{G}$ acting freely on $\widetilde{G}$ such that $\widetilde{G}/A$ is isomorphic to G. We denote the set of directed edges of G by $D(G)$. By e^{-1} we mean the reverse edge to an edge e. A voltage map

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is a set function ϕ from the set $D(G)$ to a finite group Γ such that $\phi(e^{-1}) = (\phi(e))^{-1}$ for all e in $D(G)$. The values of ϕ are called voltages and Γ is called the voltage group.

The voltage covering graph $G \times_\phi \Gamma$ derived from $\phi : D(G) \to \Gamma$ has as its vertex set $V(G) \times \Gamma$ and as its edge set $E(G) \times \Gamma$, so that an edge of $G \times_\phi \Gamma$ joins a vertex (u, g) to $(v, \phi(e)g)$, for $e = uv \in E(G)$ and $g \in \Gamma$. A vertex (u, g) is denoted by u_g, and an edge (e, g) by e_g. The voltage group Γ acts naturally on $G \times_\phi \Gamma$ as follows: for every $g \in \Gamma$,let $\Phi_g : G \times_\phi \Gamma \to G \times_\phi \Gamma$ denote the graph automorphism defined by $\Phi_g(v_{g'}) = v_{gg'}$ on verteces and $\Phi_g(e_{g'}) = e_{gg'}$ on edges. Then the natural map $G \times_\phi \Gamma \to (G \times_\phi \Gamma)/\Gamma \cong G$ is a $|\Gamma|$-fold regular covering projection. Gross and Tucker[1] showed that every regular covering of G arises from some voltage assignment of G.

Let T be a fixed spanning tree of G and let T^* be its cotree in G. Then we know that $|E(T^*)| = \beta(G)$ and any voltage covering graph is isomorphic to a voltage covering graph whose voltages are trivial on the edges of T. The following theorem gives an algebraic characterization of the isomorphisms between two voltage coverings whose voltages are trivial on the edges of T.

THEOREM. *(Fundamental theorem) Any two voltage coverings $G \times_\phi \Gamma$ and $G \times_\psi \Gamma$ are isomorphic if and only if there exists a permutation f in $S_{|\Gamma|}$ such that $\psi(e) = f\phi(e)f^{-1}$ for all e in $E(T^*)$.*

Note that any voltage map $\phi : D(G) \to \Gamma$ whose values on $E(T)$ is the identity of Γ can be written as $\phi = (g_1, g_2, \ldots, g_\beta) \in \Gamma \times \Gamma \times \cdots \times \Gamma \equiv \Gamma^\beta$, and any two such voltage maps $\phi = (g_1, g_2, \ldots, g_\beta)$ and $\psi = (g_1', g_2', \ldots, g_\beta')$ give rise to isomorphic voltage covering graphs, denoted by $\phi \sim \psi$, if and only if there is a permutation $f \in S_{|\Gamma|}$ such that $fg_if^{-1} = g_i'$ for each $i = 1, 2, \ldots, \beta$.

M.Hofmeister [3] counted the number of the isomorphism classes of double coverings of a given graph G. Also S.Hong and J.Kwak [4] counted the number of all isomorphism classes of regular four fold coverings of a graph. The aim of this paper is to count the number of isomorphism classes of regular $2p$–fold coverings of a given graph. Throughout in this paper,p will denote a prime number.

2. Isomorphic regular coverings with voltage group D_p

Let $D_p = < \sigma, \tau | \sigma^p = (1), \tau^2 = (1), \sigma\tau\sigma = \tau >$ be the dihedral subgroup of the symmetric group S_p where σ is a rotational symmetry of $2\pi/p$ and τ is a flip about the vertex $(p+1)/2$ and the middle of the opposite side of a regular $p-$gon with vertices $\{1, 2, \ldots, p\}$. Then the elements of D_p can be written as follow :

$$D_p = \{(1), \sigma, \sigma^2, \ldots, \sigma^{p-1}, \tau, \tau\sigma, \ldots, \tau\sigma^{p-1}\}$$

Now, let's consider the set of left translations of D_p by D_p is denoted by $L(D_p)$. Then $L(D_p)$ is isomorphic to the dihedral group D_p. The fundamental theorem says that the number of isomorphism classes of regular coverings of a given graph G with a fixed spanning tree T on which a voltage assignment is the identity in D_p is equal to the number of conjugate classes in $L(D_p) \times \cdots \times L(D_p)$ by the modular equivalence relation in S_{2p}. Hence, in order to count the number of conjugate class in $L(D_p) \times \cdots \times L(D_p)$, we start from the following lemma.

LEMMA 2.1. *(1) Let g be an element in S_p if $g\tau\sigma^i g^{-1} = \tau\sigma^i, g\tau\sigma^j g^{-1} = \tau\sigma^j$ for some i, j with $i - j \neq 0 \pmod p$. Then $g\tau^h\sigma^k g^{-1} = \tau^h\sigma^k$ for all $h = 0, 1 \quad k = 0, 1, 2, \ldots, p-1$.*

(2) There exists g_σ in S_p such that $g_\sigma \sigma g_\sigma^{-1} = \sigma, \quad g_\sigma \tau g_\sigma^{-1} = \tau\sigma$.

(3) There exist g_{τ_i} in S_p such that $g_{\tau_i}\tau g_{\tau_i}^{-1} = \tau, \quad g_{\tau_i}\sigma g_{\tau_i}^{-1} = \sigma^i$ for $i = 1, 2, \ldots, p-1$.

Proof. (1) We first observe that

$$\begin{aligned} g\sigma^{j-i}g^{-1} &= g\sigma^j\sigma^{-i}g^{-1} \\ &= g\sigma^{-i}\sigma^j g^{-1} \\ &= g\tau\sigma^i\tau\sigma^j g^{-1} \\ &= g\tau\sigma^i g^{-1} g\tau\sigma^j g^{-1} \\ &= \tau\sigma^i\tau\sigma^j \\ &= \sigma^{j-i}. \end{aligned}$$

Since $j - i \neq 0 \pmod p$, $g\sigma g^{-1} = \sigma$. Moreover,

$$\begin{aligned} g\tau g^{-1} &= g\tau\sigma^i\sigma^{-i}g^{-1} \\ &= g\tau\sigma^i g^{-1}\sigma^{-i} \\ &= \tau\sigma^i\sigma^{-i} \\ &= \tau. \end{aligned}$$

(2) Let $g_\sigma = \sigma^{(p-1)/2}$, then $g_\sigma \sigma g_\sigma^{-1} = \sigma$ and

$$\begin{aligned} g_\sigma \tau g_\sigma^{-1} &= \sigma^{(p-1)/2}\tau\sigma^{-(p-1)/2} \\ &= \tau\sigma^{-(p-1)} \\ &= \tau\sigma. \end{aligned}$$

(3) Let $\sigma^i = (i_1, i_2, \dots, i_{(p+1)/2}, \dots, i_p)$ be the cycle of the i−th power of σ so that the $(p+1)/2$-th number $i_{(p+1)/2}$ is $(p+1)/2$. and let g_{τ_i} be the element of S_p defined by the following ;

$$g_{\tau_i} = \begin{pmatrix} 1 & 2 & \dots & (p+1)/2 & \dots & p \\ i_1 & i_2 & \dots & i_{(p+1)/2} & \dots & i_p \end{pmatrix}$$

then $g_{\tau_i}\sigma g_{\tau_i}^{-1} = \sigma^i$ and $g_{\tau_i}\tau g_{\tau_i}^{-1} = \tau$.

Indeed, if $k \in I_p = \{1, 2, \dots, p\}$, then there exists integer $m (1 \leq m \leq p)$ such that $k = (p+1)/2 + i(m - (p+1)/2) \pmod p$. Hence we have

$$\begin{aligned} g_{\tau_i}\tau g_{\tau_i}^{-1}(k) &= g_{\tau_i}\tau(m) \\ &= g_{\tau_i}(p - m + 1) \\ &= (p+1)/2 + i(p - m + 1 - (p+1)/2) \\ &= (p+1)/2 - i(m - (p+1)/2) \\ &= (p+1)/2 + (p+1)/2 - k \\ &= p + 1 - k \\ &= \tau(k). \end{aligned}$$

THEOREM 2.2. *(1) For each $i \in I_p$, there exists $g_{\sigma_i} \in S_p$ such that $g_{\sigma^i}\sigma g_{\sigma^i}^{-1} = \sigma$ and $g_{\sigma^i}\tau g_{\sigma^i}^{-1} = \tau\sigma^i$.*

(2) For each $i \in I_p, j \in I_{p-1}$, there exists $g \in S_p$ such that $g\tau\sigma^i g^{-1} = \tau\sigma^i$ and $g\sigma g^{-1} = \sigma^j$.

(3) For each $i \neq j \in I_p$, there exists $g_{ij} \in S_p$ such that $g_{ij}\tau\sigma^i g_{ij}^{-1} = \tau\sigma^i$ and $g_{ij}\tau\sigma^j g_{ij}^{-1} = \tau\sigma^l$ for $l \neq i$.

Proof. (1) Let g_σ be the element in [Lemma 2.1] and let $g_{\sigma^i} = g_\sigma^i$. Then $g_{\sigma^i}\sigma g_{\sigma^i}^{-1} = \sigma$
and

$$\begin{aligned} g_{\sigma^i}\tau g_{\sigma^i}^{-1} &= g_\sigma^i \tau g_\sigma^{-i} \\ &= g_\sigma^{i-1}\tau\sigma g_\sigma^{-(i-1)} \\ &= g_\sigma^{i-1}\tau g_\sigma^{-(i-1)}\sigma \\ &= \dots \\ &= \tau\sigma^i. \end{aligned}$$

(2) Let $k \in I_p$ such that $ij + k = i \pmod p$ and let $g = g_{\sigma^k} g_{\tau_j}$.
Then

$$\begin{aligned} g\tau\sigma^i g^{-1} &= g_{\sigma^k} g_{\tau_j}\tau\sigma^i g_{\tau_j}^{-1} g_{\sigma^k}^{-1} \\ &= g_{\sigma^k}\tau\sigma^{ij} g_{\sigma^k}^{-1} \\ &= g_{\sigma^k}\tau g_{\sigma^k}^{-1}\sigma^{ij} \\ &= \tau\sigma^k\sigma^{ij} \\ &= \tau\sigma^{k+ij} \\ &= \tau\sigma^i \end{aligned}$$

and

$$\begin{aligned} g\sigma g^{-1} &= g_{\sigma^k} g_{\tau_j}\sigma g_{\tau_j}^{-1} g_{\sigma^k}^{-1} \\ &= g_{\sigma^k}\sigma^j g_{\sigma^k}^{-1} \\ &= \sigma^j. \end{aligned}$$

(3) Take g such that $g\tau\sigma^i g^{-1} = \tau\sigma^i$ and $g\sigma^{j-i}g^{-1} = \sigma^{l-i}$ (by 2).

Then

$$\begin{aligned} g\tau\sigma^j g^{-1} &= g\tau\sigma^i\sigma^{j-i}g^{-1} \\ &= \tau\sigma^i\sigma^{l-i} \\ &= \tau\sigma^l. \end{aligned}$$

Thus such g is the desired g_{ij} which completes the proof.

Let $\alpha_k(\beta(G); D_p)$ denote the number of conjugate classes of elements in the set $A_k = \{\phi | \phi : D(G) \to D_p$ is a voltage assignment which is nontrivial on fixed k edges of cotree $T^*\}$.

LEMMA 2.3. $\alpha_k(\beta(G); D_p) = (p-1)^{k-1} + 1 + ((2p-1)^k - (p-1)^k - p)/p(p-1)$.

Proof. Let ϕ be a voltage assignment in A_k. Lemma 2.1.(3) and Theorem 2.2.(1) say that $\sigma, \sigma^2, \ldots, \sigma^{p-1}$ are in the same conjugate class in S_p and $\tau, \tau\sigma, \ldots, \tau\sigma^{p-1}$ are also in the same conjugate class in S_p. Hence, the proof we shall give is quite elementary. It will be preceded by the following three cases.

case 1. If $\phi(e) \in \{\sigma^1, \sigma^2, \ldots, \sigma^{p-1}\}$ for each $e \in E(T^*)$ then, clearly, $|\{\psi \in A_k | \phi \sim \psi\}| = p-1$.

case 2. If ϕ is constant and $\phi(e) \in \{\tau, \tau\sigma, \ldots, \tau\sigma^{p-1}\}$ for each $e \in E(T^*)$ then $|\{\psi \in A_k | \phi \sim \psi\}| = p$ by (1) in theorem 2.2.

case 3. If ϕ is nonconstant and $\phi(e)$ cannot have only $\sigma, \sigma^2, \ldots, \sigma^{p-1}$ for each $e \in E(T^*)$, then $|\{\psi \in A_k | \phi \sim \psi\}| = p(p-1)$ by mixing (1),(2),(3) in theorem 2.2.

Conseguently, The result comes from these three cases.

Now we consider the set of left translation $L(D_p)$ of D_p. Since $L(D_p)$ is isomorphic to D_p, we can write $L(D_p)$ as follow :

$$L(D_p) = \{\overline{(1)}, \overline{\sigma}, \overline{\sigma^2}, \ldots, \overline{\sigma^{p-1}}, \overline{\tau}, \overline{\tau\sigma}, \ldots, \overline{\tau\sigma^{p-1}}\}.$$

Now,we give our main result.

THEOREM 2.4. *The number of isomorphism classes of regular coverings of a graph G with voltage group $\Gamma = D_p$ is $\{(p-1)p^\beta + p(p-2)2^\beta + (2p)^\beta\}/p(p-1)$, where β is the Betti number of G.*

Proof. Let Iso($G : D_p$) be the number of isomorphism classes of regular coverings of G whose voltage group is D_p. Once we have chosen k positions for $\overline{\sigma}, \overline{\sigma^2}, \ldots, \overline{\sigma^{p-1}}, \overline{\tau}, \overline{\tau\sigma}, \overline{\tau\sigma^2}, \ldots, \overline{\tau\sigma^{p-1}}$,then the other $(\beta - k)$ positions will be filled up with identity $\overline{(1)}$.

By theorem 4 in [5] and $L(D_p) \simeq D_p$, we have

$$\begin{aligned} Iso(G : D_p) &= | \quad L(D_p) \times \cdots \times L(D_p) \quad / \sim \ | \\ &= 1 + \sum_{k=1}^{\beta} \binom{\beta}{k} \alpha_k(\beta(G) : D_p). \end{aligned}$$

Thus
$p(p-1)Iso(G : D_p) = p(p-1) + (p-1)(p^\beta - 1) + p(p-2)(2^\beta - 1) + (2^\beta p^\beta - 1)$.
Dividing both sides by $p(p-1)$, we have our theorem.

Since $D_3 = S_3$, we have the following Theorem 6 in [5].

COROLLARY 2.5. *Iso*$(G : S_3) = 2^{\beta-1} + 3^{\beta-1} + 6^{\beta-1}$.

3. Regular 2p-fold covering graphs

Let Γ be a finite group of order $2p$. Then Γ is isomorphic to

$$\begin{cases} D_p & \text{if } \Gamma \text{ is nonabelian and } p \neq 2, \\ Z_{2p} & \text{if } \Gamma \text{ is abelian and } p \neq 2, \\ Z_4 & \text{if } \Gamma \text{ is cyclic and } p = 2, \\ Z_2 \times Z_2 & \text{if } \Gamma \text{ is not cyclic and } p = 2. \end{cases}$$

We now introduce some well-known results. Let Γ be a finite group and let Iso($G : \Gamma$) be the number of isomorphism classes of regular $|\Gamma|$-fold covering graphs of a given graph G whose voltage group is Γ.

THEOREM 3.1. *([5])*
(1)Iso $(G : Z_p) = (p^\beta + p - 2)/(p-1)$ *if p is a prime,*
(2)Iso $(G : Z_p \times Z_q) =$ *Iso* $(G : Z_p)$ *Iso* $(G : Z_q)$ *if p and q are distinct primes,*

(3)Iso $(G : Z_4) = 2^{\beta-1}(1+2^\beta)$.

THEOREM 3.2. *([4]) Iso*$(G : Z_2 \times Z_2) = (2 \times 4^{\beta-1}+1)/3 + 2^{\beta-1}$.

Let Iso(G:n) be the number of isomorphism classes of regular n-fold covering graphs of given graph G. Summarizing the above theorems and the theorem 2.4, we get our main result.

THEOREM 3.3. *(1)Iso*$(G : 4) = (2 \times 4^{\beta-1}+1)/3 + 2^{2\beta-1}$,
(2)Iso$(G : 2p) = 2 + (p^\beta - 1)/p + (p-2)(2^\beta - 1)/(p-1) + (2^\beta p^\beta - 1)/p(p-1)$
$+(2^\beta p^\beta - 2^\beta - p^\beta - p + 2)/(p-1)$ *if* $p \neq 2$.

Proof. Let Γ_1 and Γ_2 be two isomorphic groups. and let $L(\Gamma_1)$ and $L(\Gamma_2)$ the left translation groups of Γ_1 and Γ_2 respertively. Then $L(\Gamma_1)$ and $L(\Gamma_2)$ are conjugate subgroups of $S_{|\Gamma_1|} = S_{|\Gamma_2|}$.

Let ϕ be an voltage assignment and let $L(\phi)$ be the corresponding voltage assignment. Then $G \times_\phi \Gamma$ and $G \times_{L(\phi)} L(\Gamma)$ are isomorphic coverings of G. Then we have the followings.

(1) Iso(G:4) = Iso$(G : Z_4)$ + Iso$(G : Z_2 \times Z_2)$ − Iso$(G : Z_2)$,

(2) Iso$(G : 2p)$ = Iso$(G : D_p)$ + Iso$(G : Z_2 \times Z_p)$ − Iso$(G : Z_p)$ − Iso$(G : Z_2)$ + 1
if $p \neq 2$.

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