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ON HIGHER DIMENSIONAL η-FUNCTIONS

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For $p \geq 2$, a *p-link* means a two component link (M, K) in S^{2p+1}, where M is a p-sphere and K a $(2p-1)$-sphere. For a p-link $L = (M, K)$, we define its η-function as follows. Since $\pi_1 M = 0$, there exists a lift $\widetilde{M}$ of M in the infinite cyclic cover $\widetilde{X}$ of $S^{2p+1} \setminus K$. Let ℓ be a zero-push-off of M in S^{2p+1} and $\tilde{\ell}$ its lift in $\widetilde{X}$. Denote by t the generator of the covering transformation corresponding to a meridian of K. Let $A(t) \in \mathbf{Z}[t, t^{-1}]$ be an annihilator of the class $[\tilde{\ell}] \in H_p(\widetilde{X})$. Then $A(t)\tilde{\ell}$ is a boundary of a $(p+1)$-chain ζ in $\widetilde{X}$. Define

$$\eta_L(t) = \frac{\sum_{i=-\infty}^{\infty} \mathrm{Int}(\zeta, t^i \widetilde{M}) t^i}{A(t)},$$

where Int(,) is the usual intersection number. As in the classical dimensional case, η_L is well defined and invariant under I-equivalence of p-links [**KY**].

In the case $p = 1$, the η-function can be described in terms of the Alexander polynomials of the link and its components [**J**].

PROPOSITION 1. *Let L be a p-link for some $p \geq 2$. Then*

(a) $\eta_L(t^{-1}) = (-1)^{p+1} \eta_L(t)$,

(b) $\eta_L(1) = 0$,

(c) *$\eta_L(t)$ is of the form $f(t)/g(t)$ where $f(t)$, $g(t) \in \mathbf{Z}[t, t^{-1}]$ and $|g(1)| = 1$.*

Proof. (a) Let $L = (M, K)$. Choose a zero-push-off ℓ of M. Then $L' = (\ell, K)$ is ambient isotopic to L. Let ζ be as above and ζ' be a

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$(p+1)$-chain such that $\partial\zeta' = A(t)\widetilde{M}$. Then

$$\begin{aligned}
\eta_L(t^{-1}) &= \frac{\sum_{i=-\infty}^{\infty} \operatorname{Int}(\zeta, t^i\widetilde{M})t^{-i}}{A(t^{-1})} \\
&= \frac{\sum_{i=-\infty}^{\infty} A(t)\operatorname{Int}(\zeta, t^i\widetilde{M})t^{-i}}{A(t)A(t^{-1})} \\
&= \frac{\sum_{i=-\infty}^{\infty} \operatorname{Int}(\zeta, t^iA(t)\widetilde{M})t^{-i}}{A(t)A(t^{-1})} \\
&= \frac{\sum_{j=-\infty}^{\infty} \operatorname{Int}(t^j\zeta, \partial\zeta')t^{j}}{A(t)A(t^{-1})} \\
&= \frac{\sum_{j=-\infty}^{\infty} (-1)^{p+1}\operatorname{Int}(t^j\partial\zeta, \zeta')t^{j}}{A(t)A(t^{-1})} \\
&= \frac{\sum_{j=-\infty}^{\infty} (-1)^{p+1}(-1)^{p(p+1)}\operatorname{Int}(\zeta', t^jA(t)\tilde{\ell})t^{j}}{A(t)A(t^{-1})} \\
&= \frac{\sum_{j=-\infty}^{\infty} (-1)^{p+1}A(t^{-1})\operatorname{Int}(\zeta', t^j\tilde{\ell})t^{j}}{A(t)A(t^{-1})} \\
&= (-1)^{p+1}\eta_{L'}(t) \\
&= (-1)^{p+1}\eta_L(t).
\end{aligned}$$

(b) It follows from the fact that $lk(\ell, M) = 0$.

(c) As in the proof of [**L2**, Corollary 1.3], there exists an annihilator $\Delta(t) \in \mathbf{Z}[t, t^{-1}]$ of $H_p(\widetilde{X})$, satisfying $\Delta(1) = \pm 1$.

THEOREM 2. *Let $\eta(t) = f(t)/g(t)$ with $f(t)$, $g(t) \in \mathbf{Z}[t, t^{-1}]$. Suppose that*

(a) $\eta(t^{-1}) = (-1)^{p+1}\eta(t)$

(b) $\eta(1) = 0$

(c) $|g(1)| = 1$

for some integer $p \geq 2$. Then there exists a p-link L such that $\eta_L(t) = \eta(t)$.

Proof. We use Levine's construction in [**L1**]. By multiplying $g(t^{-1})$ both on the denominator and the numerator of $\eta(t)$, we may assume that

$\eta(t)$ is of the form

$$\frac{F(t)+(-1)^{p+1}F(t^{-1})}{G(t)}$$

where $G(t^{-1})=G(t)$ and $G(1)=1$. Let K_0 be a $(2p-1)$-dimensional unknot in $\mathbf{S}^{2p+1}$. Let $S=(S_{-m},S_{-m+1},\dots,S_m)$ be a trivial link of p-spheres in a ball $B\subset \mathbf{S}^{2p+1}\setminus K_0$ and S_a be a p-dimensional unknot in B disjoint from S such that

$$lk(S_a,S_i)=g_i$$

where $G(t)=\sum_{i=-m}^{m} g_i t^i$. Let

$$S_b=S_{-m}\sharp S_{-m+1}\sharp\cdots\sharp S_m$$

be an oriented connected sum, in which the connected sum between S_i and S_{i+1} is taken along an arc A_i connecting S_i and S_{i+1}, oriented from S_i to S_{i+1} and going once around $\mathbf{S}^{2p+1}\setminus K_0$ in a fixed direction for each $-m\le i\le m-1$. A spherical modification on the trivial normal framings on S_a and S_b will make K_0 into a knot K in a new sphere $\widetilde{\mathbf{S}^{2p+1}}$. Let $\widetilde{X}$ be the infinite cyclic cover of $\mathbf{S}^{2p+1}\setminus K$. Then $H_p(\widetilde{X})$ is generated by $\tilde{\alpha}$ and $\tilde{\beta}$ over $\mathbf{Z}[t,t^{-1}]$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are lifts of some meridian spheres α and β to S_a and S_b, respectively. A computation shows that

$$\begin{bmatrix} 0 & G(t) \\ (-1)^{p+1}G(t) & 0 \end{bmatrix}$$

is a presentation matrix for $H_p(\widetilde{X})$.

Let $F(t)=\sum_{i=-n}^{n} f_i t^i$ and let $\mathcal{M}=(\mathcal{M}_{-n},\mathcal{M}_{-n+1},\dots,\mathcal{M}_n)$ be another trivial link of p-spheres in B disjoint from $S_a\cup S_b$ such that

$$lk(\mathcal{M}_i,S_a)=f_i \qquad \text{for all } i$$
$$lk(\mathcal{M}_i,S_j)=\begin{cases}1 & \text{for } i=j=0\\ 0 & \text{otherwise.}\end{cases}$$

Let $M_0=\mathcal{M}_{-n}\sharp\mathcal{M}_{-n+1}\sharp\cdots\sharp\mathcal{M}_n$ be an oriented connected sum obtained in a similar manner as S_b. Passing to the same spherical modification, M_0 becomes M such that a lift $\widetilde{M}$ of M in $\widetilde{X}$ is homologous to $F(t)\tilde{\alpha}+\tilde{\beta}$. Let L be the p-link (M,K). A computation shows that

$$\eta_L(t) = \frac{F(t) + (-1)^{p+1}F(t^{-1})}{G(t)}.$$

Given a p-link $L = (M, K)$, suppose that M is null-homotopic in $\mathbf{S}^{2p+1} \setminus K$. Then there exists a map $g : \mathbf{D}^{p+1} \to \mathbf{S}^{2p+1} \setminus K$ such that

$$\begin{array}{ccc} \mathbf{S}^p & \equiv & M \\ \cap & & \cap \\ \mathbf{D}^{p+1} & \xrightarrow{g} & \mathbf{S}^{2p+1} \setminus K \end{array}$$

commutes. Let $g' : \mathbf{D}^{p+1} \to (\mathbf{S}^{2p+1} \setminus K) \times [0,1]$ be defined by

$$g'(x) = \left(g(x), \frac{1 - \|x\|}{2}\right) \quad \text{for all} \quad x \in \mathbf{D}^{p+1}.$$

We may assume that g' is a self-transverse immersion. Let $\widetilde{X}$ be the infinite cyclic cover of $\mathbf{S}^{2p+1} \setminus K$ and let $\tilde{g}$ and $\tilde{g}'$ be lifts of g and g', respectively, such that

$$\begin{array}{ccccc} \mathbf{S}^p & \subset & \mathbf{D}^{p+1} & \xrightarrow{\tilde{g}'} & \widetilde{X} \times [0,1] \\ \| & & & & \cup \\ \mathbf{S}^p & \subset & \mathbf{D}^{p+1} & \xrightarrow{\tilde{g} \times \{0\}} & \widetilde{X} \times \{0\} \end{array}$$

commutes. Define

$$\sigma_L(t) = \sum_{i=1}^{\infty} \operatorname{Int}(t^n \tilde{g}'(\mathbf{D}^{p+1}), \tilde{g}'(\mathbf{D}^{p+1}))(t^n - 1).$$

According to the next theorem, σ_L is independent of the choice of the map g. If K is null-homotopic in $\mathbf{S}^{2p+1} \setminus M$ also, then the map g 'extends' to a link-map $g_L : \mathbf{S}^{p+1} \cup \mathbf{S}^{2p} \to \mathbf{S}^{2p+2}$ whose restriction to equators represents the p-link L. Then $\sigma_L(t)$ is equal to Krik's σ-invariant $\sigma_{g_L}(t)$ [K].

THEOREM 3. *Suppose $L = (M, K)$ is a p-link such that M is null-homotopic in $\mathbf{S}^{2p+1} \setminus K$. Then*

$$\eta_L(t) = \sigma_L(t) + (-1)^{p+1}\sigma_L(t^{-1}).$$

Proof. Let g, g', $\tilde{g}$ and $\tilde{g}'$ be as above. Then, for $i \geq 1$,

$$\begin{aligned} &\operatorname{Int}(\tilde{g}(\mathbf{D}^{p+1}), t^i\tilde{g}(\mathbf{S}^p)) && \text{in } \widetilde{X} \equiv \widetilde{X} \times \{0\} \\ =&(-1)^{p+1} \operatorname{Int}(\tilde{g}'(\mathbf{D}^{p+1}), t^i\tilde{g}'(\mathbf{D}^{p+1})) && \text{in } \widetilde{X} \times [0,1] \\ =&\operatorname{Int}(t^i\tilde{g}'(\mathbf{D}^{p+1}), \tilde{g}'(\mathbf{D}^{p+1})). \end{aligned}$$

By Proposition 1, we have

$$\begin{aligned} &\eta_L(t) \\ =&\sum_{i=1}^{\infty} \operatorname{Int}(\tilde{g}(\mathbf{D}^{p+1}), t^i\tilde{g}(\mathbf{S}^p))((t^i - 1) + (-1)^{p+1}(t^{-i} - 1)) \\ =&\sigma_L(t) + (-1)^{p+1}\sigma_L(t^{-1}). \end{aligned}$$

Let $L = (M, K)$ be a p-link with K unknotted in $\mathbf{S}^{2p+1}$. Then there exists an embedding $f_L : \mathbf{S}^p \hookrightarrow \mathbf{S}^1 \times \mathbf{D}^{2p}$ making

$$\begin{array}{ccc} \mathbf{S}^p & \hookrightarrow & \mathbf{S}^1 \times \mathbf{D}^{2p} \\ ||| & & ||| \\ M & \hookrightarrow & \mathbf{S}^{2p+1} \setminus K \end{array}$$

commutative. Let $I_L = (I_1, I_2, \dots)$ be equal to the Hacon invariant

$$I_{f_L} \in \bigoplus_{i=1}^{\infty} \mathbf{Z}$$

of the embedding f_L [**Ha**]. Then obviously I_L is well defined. Identify

$$\bigoplus_{i=1}^{\infty} \mathbf{Z} \equiv \bigoplus_{i=1}^{\infty} \mathbf{Z}(t^i - 1) \subset \mathbf{Z}[t].$$

Then

$$I_L = I_L(t) = \sum_{i=1}^{\infty} I_i(t^i - 1).$$

THEOREM 4. *Let $L = (M, K)$ be a p-link such that K is unknotted in $\mathbf{S}^{2p+1}$. Then*

$$\eta_L(t) = I_L(t) + (-1)^{p+1} I_L(t^{-1}).$$

Proof. Let ℓ, $\tilde{\ell}$, $\widetilde{M}$, ζ and ζ' be as in the proof of Proposition 1. Since K is unknotted, $H_p(\widetilde{X}) = 0$. Therefore we may choose $A(t) = 1$ and then $\partial\zeta = \tilde{\ell}$ and $\partial\zeta' = \widetilde{M}$. Since the infinite cyclic cover of $\mathbf{S}^{2p+1} \setminus K$ is homeomorphic to $\mathbf{R}^{2p+1}$, we have, for $i > 0$,

$$\mathrm{Int}(\zeta, t^i\widetilde{M}) = lk(\tilde{\ell}, t^i\widetilde{M}) = lk(\widetilde{M}, t^i\widetilde{M})$$

and

$$\begin{aligned}\mathrm{Int}(\zeta, t^{-i}\widetilde{M}) &= (-1)^{p(p+1)}\,\mathrm{Int}(t^{-i}\widetilde{M}, \zeta)\\ &= \mathrm{Int}(\widetilde{M}, t^i\zeta)\\ &= (-1)^{p+1}\,\mathrm{Int}(\zeta', t^i\tilde{\ell})\\ &= (-1)^{p+1} lk(\widetilde{M}, t^i\tilde{\ell})\\ &= (-1)^{p+1} lk(\widetilde{M}, t^i\widetilde{M}).\end{aligned}$$

Therefore, by Proposition 1 (b), we have

$$\begin{aligned}\eta_L(t) &= \sum_{i=-\infty}^{\infty} \mathrm{Int}(\zeta, t^i\widetilde{M})t^i\\ &= \sum_{i\neq 0} \mathrm{Int}(\zeta, t^i\widetilde{M})(t^i - 1)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \operatorname{Int}(\zeta, t^i\widetilde{M})(t^i - 1) + \sum_{i=1}^{\infty} \operatorname{Int}(\zeta, t^{-i}\widetilde{M})(t^{-i}, 1) \\
&= \sum_{i=1}^{\infty} lk(\widetilde{M}, t^i\widetilde{M})(t^i - 1) + (-1)^{p+1} \sum_{i=1}^{\infty} \operatorname{Int}(\widetilde{M}, t^i\widetilde{M})(t^{-i} - 1) \\
&= \sum_{i=1}^{\infty} I_i(t^i - 1) + (-1)^{p+1} \sum_{i=1}^{\infty} I_i(t^{-i} - 1) \\
&= I_L(t) + (-1)^{p+1} I_L(t^{-1}).
\end{aligned}
$$

By Theorem 3 and Theorem 4, we obtain

COROLLARY 5. *Let $L = (M, K)$ be a p-link such that K is unknotted. Then*

$$\sigma_L(t) = I_L(t).$$

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