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A CONJUGACY THEOREM IN PRO-ČERNIKOV GROUPS *

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1. Introduction

In 1979, in their attempt to generalize a result of G.Higman [3] Losey and Stonehewer proved the following theorem [4]:

THEOREM 1.1. *Let G be a finite solvable group. Let U and V, subgroups of G, be p-conjugate for every prime p. Suppose that U and V have a common nilpotent normal supplement X (that is, $G = UX = VX$ and X is nilpotent and normal) in G and that one of the following conditions is satisfied:*

1) *X is abelian*
2) *G/X is nilpotent, or*
3) *the Sylow p-subgroups of G have class at most 2 for every prime p.*

Then U and V are conjugate.

Recently, it was shown that Losey-Stonehewer theorem holds without solvability [2]. It is desired to try to generalize the theorem for infinite groups. In fact we have some generalizations for specific classes of locally finite groups [5]. In [6] we have proved the similar theorem for profinite groups. The notations are standard. In particular, by $H \leq_c G$ we mean H is a closed subgroup of G. And $\mathrm{Syl}_p G$ denotes the set of all Sylow p-subgroups of G.

2. Pro-Černikov groups

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By a separating filter base $\mathcal{N}$ of a group G we shall mean a set of normal subgroups satisfying:

(1) If $N \in \mathcal{N}$, G/N is a Černikov group.
(2) If $L, M \in \mathcal{N}$ there exists $N \in \mathcal{N}$ such that $N \leq L \cap M$
(3) $\cap\{N : N \in \mathcal{N}\} = 1$.

Thus G possesses a separating filter base if and only if G is a residually Černikov group. We shall call such G a co-Černikov group relative to $\mathcal{N}$ and regard G as a topological space with

$$\{Hx : x \in G \text{ and there exists } N \in \mathcal{N} \text{ such that } N \leq H \leq G\}$$

as a closed sub-base.

An inverse limit of Černikov groups is called a pro-Černikov group. It is well known that compact co-Černikov groups are precisely the pro-Černikov groups [1].

The proof of the following lemma can be found in [1].

LEMMA 2.1. *Let G be a co-Černikov group with a separating filter base $\mathcal{N}$.*

(1) *If $H \leq G$, then its closure $\overline{H}$ is given by $\overline{H} = \cap\{HN : N \in \mathcal{N}\}$*
(2) *If H is a closed subgroup of G, then its normalizer $N_G(H)$ is a closed subgroup of G.*

A subgroup P of a co-Černikov group $(G, \mathcal{N})$ is called a generalized p-group if G/N is a p-group for all $N \leq_c G$ with G/N Černikov.

A subgroup P of a co-Černikov group $(G, \mathcal{N})$ will be called a generalized Sylow p-subgroup of $(G, \mathcal{N})$ if

(1) $p \leq_c G$
(2) $PN/N \in \mathrm{Syl}_p G/N$ for all $N \leq_c G$ with G/N Černikov.

THEOREM 2.2. *Let G be a pro-Černikov group.*

(1) *G possesses generalized Sylow p-subgroups for each prime p.*
(2) *The generalized Sylow p-subgroups are conjugate.*
(3) *A generalized p-group of G is contained in some generalized Sylow p-subgroup of G.*

Proof. See [1].

Let U and V be subgroups of a pro-Černikov group G. We say that U and V are p-conjugate if a generalized Sylow p-subgroup of U is conjugate to a generalized Sylow p-subgroup of V.

For the completeness we include the following theorem in [5]

THEOREM 2.3. *Let G be a Černikov group. Let U and V, subgroups of G, be p-conjugate for every prime p. Suppose that U and V have a common locally nilpotent nomal supplement X in G and that one of the following conditions is satisfied:*

(1) *X is abelian;*
(2) *G/X is locally nilpotent;*
(3) *the Sylow p-subgroups of G have class at most 2 for every prime p.*

Then U and V are conjugate.

3. Main theorem

We prove :

THEOREM 3.1. *Let G be a pro-Černikov group. Let U and V, subgroups of G, be closed and p-conjugate for every prime p. Suppose that U and V have a common closed nilpotent normal supplement X in G and that one of the following conditions is satisfied:*

(1) *X is abelian;*
(2) *G/X is nilpotent;*
(3) *the generalized Sylow p-subgroups of G have class at most 2 for every prime p.*

Then U and V are conjugate.

Proof. Let $\mathcal{N} = \{N_\alpha\}$ be a separating filter base of G. For each α, consider the canonical homomorphism

$$\phi_\alpha : G \to G/N_\alpha.$$

Since G/N_α is Černikov, and all the conditions in this theorem are preserved by factoring, we can apply Theorem 2.3 to

$$G/N_\alpha = UN_\alpha/N_\alpha \cdot XN_\alpha/N_\alpha = VN_\alpha/N_\alpha \cdot XN_\alpha/N_\alpha$$

to obtain an element $x_\alpha N_\alpha \in G/N_\alpha$ such that $(UN_\alpha/N_\alpha)^{x_\alpha N_\alpha} = VN_\alpha/N_\alpha$. This means that $(UN_\alpha)^{x_\alpha} = VN_\alpha$. If we let $F_\alpha = \{x \in G | (UN_\alpha)^x = VN_\alpha\}$, then F_α is non-empty and $F_\alpha = x_\alpha N_G(U)$ for some $x_\alpha \in F_\alpha$. By Lemma 2.1, F_α is closed. Now it is clear that $F_\alpha \subseteq F_\beta$ if $N_\alpha \subseteq N_\beta$. So $\{F_\alpha\}$ has the finite intersection property. The compactness of G implies that

$$\bigcap_\alpha F_\alpha \neq \phi$$

If we takes $x \in \cap_\alpha F_\alpha$. Then

$$V = \overline{V} = \cap\{VN_\alpha\} = \cap\{(UN_\alpha)^x\} = (\cap\{UN_\alpha\})^x = \overline{U}^x = U^x.$$

Therefore, U and V are conjugate.

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