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UNBOUNDED DERIVATIONS ON COMPACT ACTIONS OF C^*-ALGEBRAS

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1. Introduction

Let A be a C^*-algebra and let $Aut(A)$ be the group of $*$-automorphisms of A. Let G be a locally compact group. Let $\alpha : G \to Aut(A)$ be a strongly continuous homomorphism. The triple (A, G, α) is called a C^*-dynamical system. Let δ be a densely defined *-derivation on a C^*-algebra A. We say that δ is a generator of a C^*-dynamical system $(A, \mathbf{R}, \alpha)$ with real number group $\mathbf{R}$ if there exists a strongly continuous one-parameter group $\alpha : \mathbf{R} \to Aut(A)$ of *-automorphisms such that δ is the infinitesimal generator for α, i.e. $\delta(x) = \lim_{t\to 0} \frac{\alpha_t(x)-x}{t}$ for all x in the domain of δ. A derivation δ is called a pre-generator if there exists an extension δ' of δ which is a generator of a C^*-dynamical system $(A, \mathbf{R}, \alpha)$. Bratteli, Goodman and Jorgensen [4,5 etc] showed that unbounded derivation tangential to compact group of automorphisms with some domain condition is automatically a generator.

In this paper we prove that a densely defined closed *-derivation commuting with α on A is a generator if the C^*- dynamical system (A, G, α) satisfies certain conditions.

2. Preliminaries

Let (A, G, α) be a C^*-dynamical system and let G be a locally compact abelian group. Let dg denote the Haar measure on G. Let $\widehat{G}$ be the dual

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group of G. Let $L^1(G)$ be the group algebra of G. For $f \in L^1(G)$ the bounded linear operator $\alpha_f : A \to A$ is defined by

$$\alpha_f(x) = \int_G f(g)\alpha_g(x)dg \quad \text{for all } x \in A.$$

The Arveson spectrum of α is defined by $Sp(\alpha) = \{\gamma \in \widehat{G} | \alpha_f = 0 \Rightarrow \hat{f}(\gamma) = 0\}$. If $x \in A$, let $Sp_\alpha(x) = \{\gamma \in \widehat{G} | \alpha_f(x) = 0 \Rightarrow \hat{f}(\gamma) = 0\}$. If E is a subset of $\widehat{G}$, the spectral subspace associated with E is defined by $A^\alpha(E)$= the norm closure of $\{x \in A : Sp_\alpha(x) \subset E\}$. Let A^α_F be the union of $A^\alpha(K)$ for all compact subsets K of $\widehat{G}$. It is known that A^α_F is a dense *-subalgebra of A [2]. For $\gamma \in \widehat{G}$ and $x \in A$, we set

$$P_\gamma(x) = \int_G \overline{\gamma(g)}\alpha_g(x)dg.$$

Then P_γ is a projection of norm 1 from A onto $A^\alpha(\gamma)$, where $A^\alpha(\gamma)$ is the spectral subspace corresponding to γ. A^α denotes the fixed point algebra of α, i.e. $A^\alpha = \{x \in A | \alpha_g(x) = x \text{ for all } g \in G\}$.

Let (A, G, α) be a C^*-dynamical system. The representation $\pi : A \to L(H)$, where $L(H)$ is the set of all bounded linear operators on a Hilbert space H, of A is α-covariant if α extends to a σ-weakly continuous action $\widehat{\alpha}$ of G in the σ-weak closure $\pi(A)''$ of $\pi(A)$ such that $\widehat{\alpha}_g(\pi(x)) = \pi(\alpha_g(x))$ for all $x \in A$. If G is compact, there exists always a nondegenerately α-covariant faithful representation of (A, G, α). We denote von Neumann algebra $\pi(A)''$ generated by $\pi(A)$. For any non-empty subset S of $\pi(A)$, S' denotes the commutant of S in $L(H)$.

REMARK 2.1. Let (A, G, α) be a C^*-dynamical system with a compact group G. Any approximate unit of A^α is also the approximate unit of A [9]. If A is represented nondegenerately in α-covariant faithful representation, by the covariance $A^{\alpha''} = A''^{\alpha}$.

3. Main results

Let G be a compact abelian group and we assume that A is represented faithfully, covariantly and nondegenerately on a Hilbert space H. The

σ-weak closure of $A^\alpha(\gamma)A^\alpha(\gamma)^*$ is the σ-weak closed two sided ideal of $A^{\alpha''}$ for each $\gamma \in \widehat{G}$. Let $E(\gamma)$ be a projection in the center Z of $A^{\alpha''}$ such that the σ-weak closure of $A^\alpha(\gamma)A^\alpha(\gamma)^* = A^{\alpha''}E(\gamma)$. Put $C(\gamma) = A^{\alpha''}E(\gamma) \cap E(\gamma)A''E(\gamma)$. There are the linear map $\beta : C(-\gamma) \to C(\gamma)$ such that $\beta_\gamma(a)x = xa$ for all $a \in C(-\gamma)$ and $x \in A^\alpha(\gamma)$ and the closed linear map $L(\gamma)(x\xi) = \delta(x)\xi$ for all $x \in A^\alpha(\gamma)$ and $\xi \in H$. These maps were considered in [3]. A densely defined *-derivation δ is called tangential with respect to (A, G, α) if

(1) $\delta(\alpha_g(x)) = \alpha_g(\delta(x))$ for all $x \in D(\delta)$,

(2) $\delta(x) = 0$ for all $x \in A^\alpha$.

These consepts were first studied in [4,5,6].

THEOREM 3.1. *Let (A, G, α) be a C^*-dynamical system and let G be a compact abelian group. Let δ be a *-closed derivation tangential with respect to (A, G, α). Assume that $A^{\alpha'} \cap A' = \mathbf{C}I$. Then the following conditions are equivalent:*

(1) *δ is bounded,*

(2) *there exists an element h in Z such that $\delta = ad_{ih}$,*

(3) *there exists an element h in Z such that*

$$L(\gamma) = hE(\gamma) - \beta_\gamma(E(-\gamma)hE(-\gamma)) \quad \text{for all } \gamma \in \widehat{G}.$$

Proof. (1) $\Rightarrow$ (2). Since δ is a bounded derivation on A, there exists $h \in A''_+$ such that $\delta = ad_{ih}$. Since $\delta|_{A^\alpha} = 0$, h is contained in $A^{\alpha'}$. Since δ commutes with α, we have

$$\delta(\alpha_g(x)) = i(h\alpha_g(x) - \alpha_g(x)h) = i(\widehat{\alpha}_g(h)\alpha_g(x) - \alpha_g(x)\widehat{\alpha}_g(h))$$

where $\widehat{\alpha}_g$ is an extension of α_g on A''. Hence $h - \widehat{\alpha}_g(h)$ is contained in A' for all $g \in G$. Since $A^{\alpha'} \cap A' = \mathbf{C}I$, there exists a complex number λ such that $h - \widehat{\alpha}_g(h) = \lambda I$. Let $h_n = (\alpha_g)^n(h)$ and then $\|h_n\| \geq |\|h\| - n\lambda|$. If we take enough large integer n, then λ must be zero. So we have $\widehat{\alpha}_g(h) = h$ for all $g \in G$ and h is contained in $A^{\alpha''}$.

(2) $\Rightarrow$ (3). Supposed that there exists h in Z such that $\delta(x) = hx - xh$ for all $x \in D(\delta)$. Note that $xE(-\gamma) = E(\gamma)x = x$ for all $x \in A^\alpha(\gamma)$. For each $x \in A^\alpha(\gamma) \cap D(\delta)$, we have

$$\delta(x) = hx - xh = hx - xE(-\gamma)hE(-\gamma) = (hE(\gamma) - \beta_\gamma(E(-\gamma)hE(-\gamma))x \cdot$$

Put $\widehat{L(\gamma)} = hE(\gamma) - \beta_\gamma(E(-\gamma)hE(-\gamma))$. Then $L(\gamma)$ and $\widehat{L(\gamma)}$ agree on $(A^\alpha(\gamma) \cap D(\delta))H$. Since $A^\alpha(\gamma) \cap D(\delta)$ is dense in $A^\alpha(\gamma)$ and since $L(\gamma)$ is closed, $L(\gamma)$ is bounded. Thus $L(\gamma) = \widehat{L(\gamma)}$.

(3) $\Rightarrow$ (1). Since $\widehat{G}$ is discrete, for each $x \in A_F^\alpha \cap D(\delta)$, x can be written as a finite linear combination of $x_{\gamma_i} \in A^\alpha(\gamma_i)$, say $x = \sum x_{\gamma_i}$ where $x_{\gamma_i} \in A^\alpha(\gamma_i)$. Then we have for all $\xi \in H$

$$\delta(x)\xi = \sum \delta(x_{\gamma_i})\xi = \sum (L(\gamma_i)x_{\gamma_i} - x_{\gamma_i}E(-\gamma_i)hE(-\gamma_i))\xi.$$

Since $xE(-\gamma) = E(\gamma)x = x$ for all $x \in A^\alpha(\gamma)$ and $h \in Z$,

$$\sum hE(\gamma_i)x_{\gamma_i} - x_{\gamma_i}E(-\gamma_i)hE(-\gamma_i) = \sum hx_{\gamma_i} - x_{\gamma_i}h = hx - xh.$$

Therefore $\delta(x) = hx - xh$ for all $x \in A_F^\alpha \cap D(\delta)$. Since $A_F^\alpha \cap D(\delta)$ is dense in A, $\delta(x) = hx - xh$ on A. In particular δ is bounded.

THEOREM 3.2. *Let (A, G, α) be a C^*-dynamical system with a compact abelian group. Let δ be a *-derivation such that $A_F^\alpha = D(\delta)$ and $\delta(A^\alpha) \subset A_F^\alpha$. Supposed that there exists a faithful α-covariant representation $\{\pi, U, H\}$ of (A, G, α) such that $\{U_g | g \in G\}$ is contained in $\pi(A)''$. Then δ is a bounded perturbation of a derivation tangential to (A, G, α).*

Proof. Let $\{\pi, U, H\}$ be such a representation and let $(\pi(A), G, \widehat{\alpha})$ be the induced C^*-dynamical system. Let $M = \pi(A)''$. Let δ_π be the induced *-derivation on $\pi(D(\delta))$. Since π is faithful, we drop the notation π for simplicity. As in the proof of Lemma 5.1 of [5], there exists an element h in $\sum_{\gamma \in \widehat{G}} \oplus M^\alpha(\gamma)$ such that $\delta|_{\widehat{A^\alpha}} = ad_{ih}|_{\widehat{A^\alpha}}$. Define a linear map $\delta_0 : \widehat{A^\alpha} \to A$ by the formula $\delta_0 = \delta - ad_{ih}|_{\widehat{A^\alpha}}$. Then clearly $\delta_0|_{\widehat{A^\alpha}} = 0$. Since $D(\delta) = \widehat{A_F^\alpha}$, by Lemma 2.7.5 of [2] there exists an element L_0 in M such that $\delta_0(x) = L_0 x$ for each $x \in \widehat{A^\alpha}$. Since multiplication is separately continuous under σ-weak topology, $\delta_0|_{\widehat{A^\alpha}}$ can be extended to $\widehat{M^\alpha}$ and $\delta_0|_{\widehat{M^\alpha}} = 0$. Since U_g is fixed by $\widehat{\alpha}$, we have

$$\delta_0(\widehat{\alpha}_g(x)) = \delta_0(U_g x U_g^*) = U_g \delta_0(x) U_g^* = \widehat{\alpha}_g(\delta_0(x)).$$

Therefore δ_0 is tangential to the C^*-dynamical system (A, G, α).

The shift dynamical system given in [8] is a nice example of the above theorem.

EXAMPLE 3.3. Let X be a infinite compact Hausdorff space, σ a homeomorphism of X and ϕ be a map from the integer group $\mathbf{Z}$ to X such that $\phi(\mathbf{Z})$ is dense in X and $\sigma(\phi(n)) = \phi(n+1)$ for each $n \in \mathbf{Z}$. Let $(C(X), \mathbf{Z}, \alpha)$ be a C^*-dynamical system induced by (X, σ) such that $\alpha^n(f)(x) = f(\sigma^{-n}(x))$ for all $x \in X, n \in \mathbf{Z}$. And let $(C(X) \times_\alpha \mathbf{Z}, \widehat{\mathbf{Z}}, \widehat{\alpha})$ be a dual system of $(C(X), \mathbf{Z}, \alpha)$. For each $n \in \mathbf{Z}$ let $\phi(n) = x_n$ and let μ_{x_n} be a pure state of $C(X)$ defined by $\mu_{x_n}(f) = f(x_n)$ for all $f \in C(X)$. Since the isotropy group G_{x_n} of x_n is trivial, by Theorem 3.3.7 of [10] μ_{x_n} has uniquely state extension $\tilde{\mu}_{x_n}$ of $C(X) \times_\alpha \mathbf{Z}$. Let ds be the normalized Haar measure on the torus $\mathbf{T}$. If we consider the map $P_0 : C(X) \times_\alpha \mathbf{Z} \to C(X)$ defined by $P_0(x) = \int_{\mathbf{T}} \widehat{\alpha}_g(x) ds$, then we have $\tilde{\mu}_{x_n} = \mu_{x_n} \circ P_0$. Since $\tilde{\mu}_{x_n}$ is an $\widehat{\alpha}$-invariant and pure state of $C(X) \times_\alpha \mathbf{Z}$, the representation (π_n, U_n, H_n) induced by $\tilde{\mu}_{x_n}$ is a irreducible covariant representation of $(C(X) \times_\alpha \mathbf{Z}, \mathbf{T}, \widehat{\alpha})$. We consider the covariant representation $(\pi = \oplus\pi_n, U = \oplus U_n, H = \oplus H_n)$. Since $\|\tilde{\mu}_{x_n} - \tilde{\mu}_{x_m}\| = 2$ for $n \neq m$, $\{\pi_n | n \in \mathbf{Z}\}$ are pairwise disjoint. So we have $\pi(A)'' = \sum \oplus \pi_n(A)''$ by Theorem 10.3.5 of [7]. Since $\phi(\mathbf{Z})$ is dense in X, $\{\mu_{x_n} | n \in \mathbf{Z}\}$ separates the points of $C(X)$. Then π is faithful and U_s is contained in $\pi(A)''$ for each $s \in \mathbf{T}$. Hence the covariant representation (π, U, H) is our desired representation.

Let (A, G, α) be a C^*-dynamical system and let G_1 be a compact normal subgroup of G. Let A_1 be the fixed point subalgebra of A under α_{G_1}. Let dg_1 be the normalized Haar measure on G_1. Then the map defined by $P_1(x) = \int_{G_1} \alpha_g(x) dg_1$ is the conditional expectation from A onto A_1. We consider a C^*-dynamical system $(A_1, G/G_1, [\alpha]^1)$ with the action $[\alpha]^1$ on the quotient group G/G_1. A C^*-dynamical system (A, G, α) is called G-finite if the spectral subspace $A^\alpha(\gamma)$ is finite dimensional for each $\gamma \in \widehat{G}$.

THEOREM 3.4. *Let (A, G, α) be a C^*-dynamical system with a compact group G. Let $\{G_i\}_{i \in I}$ be a net of compact open subgroups of G. Let A_i be the fixed point subalgebra of A under α_{G_i}. Let δ be a densely defined closed *-derivation commuting with α. Assume that $\cup A_i$ be a* *dense *-subalgebra of A and that $(A_i, G/G_i, [\alpha]^i)$ be G/G_i-finite. Then δ is a generator of a C^*-dynamical system of A.*

Proof. Let P_i be the conditional expectation from A to A_i and let D be a domain of δ. Since P_i is norm continuous, $P_i(D)$ is dense in A_i. Put $P_i(D) = D_i$ and $\delta_i = \delta|_{D_i}$. Since δ commutes with α, δ_i also commutes with $[\alpha]^i$. By Lemma 2.5.8 of [2] $\delta_i(D_i) \subset A_i$. Let $A_i(\gamma)$ be the spectral subspace of $(A_i, G/G_i, [\alpha]^i)$ corresponding to $\gamma \in (\widehat{G/G_i})$. Let P_i^γ be a conditional expectation from A_i onto $A_i(\gamma)$ defined in the Preliminaries. Since $P_i^\gamma(D_i)$ is dense in $A_i(\gamma)$ and $A_i(\gamma)$ is finite dimensional, we have $P_i^\gamma(D_i) = A_i(\gamma)$. Since $A_i(\gamma)$ is closed, $\delta_i|_{A_i(\gamma)}$ is closed map. Since δ_i commutes with $[\alpha]^i$, $\delta_i|_{A_i(\gamma)} : A_i(\gamma) \to A_i(\gamma)$ is bounded by the closed graph theorem. Let A_i^F be the set of G/G_i-finite elements of $(A_i, G/G_i, [\alpha]^i)$. Then every element of A_i^F is the analytic element for δ_i. Futhermore by Theorem 2.6.1 in [2] δ_i is a generator of a C^*-dynamical system of A_i. Then by the Hille-Yosida theorem we have

$$\|(I - \delta_i)(x)\| \geq \|x\| \qquad \text{for all } x \in D_i.$$

Since $\cup A_i$ is a dense *-subalgebra of A, $\cup A_i^F$ is dense *-subspace of analytic elements of δ. Put $D^0 = \cup D_i$. Then D^0 is dense in A and

$$\|(I - \delta)(x)\| \geq \|x\| \qquad \text{for each } x \in D^0,$$

i.e. $\delta|_{D^0}$ is dissipative. By Lemma 3.1.14. in ([1]) $\delta|_{D^0}$ is closable and its closure is also dissipative. Let δ' be the closure of $\delta|_{D^0}$. Then by the Hille-Yosida theorem δ' is the generator of a C^*-dynamical system of A. Since δ is closed, we have $\delta' \subset \delta$. Since $\cup A_i^F \subset A_\delta$ where A_δ is the set of all analytic elements of δ and

$$A = (I - \delta')(D(\delta')) \subset (I - \delta)(D(\delta)),$$

δ is also a generator of a C^*-dynamical system of A.

Let G be a locally compact abelian group and let $\widehat{G}$ be its dual group. For any arbitrary nonvoid subset H of G let $A(H)$ denote the subset of $\widehat{G}$ consisting of all γ in $\widehat{G}$ such that $\gamma(H) = 1$ and called *annihilator* of H in $\widehat{G}$.

LEMMA 3.5. *Let (A, G, α) be a C^*-dynamical system. Let G be a locally compact abelian group and let e be its unit. $\{G_i\}_{i\in I}$ be the directed system of compact open subgroups of G such that $\cap_{i\in I}G_i = \{e\}$ and A_i be the fixed point algebra under α_{G_i}. Then $\cup A_i$ is a dense *-subalgebra of A.*

Proof. Let K be a compact subset of $\widehat{G}$. Supposed that K is contained in the annihilator $A(G_i)$ in $\widehat{G}$ for some $i \in I$. Since $A(G_i)$ is open, we can choose a neighborhood U_K of K in $A(G_i)$. Choose $f \in L^1(G)$ such that $\hat{f} = 1$ on U_K. Since $\alpha_f(x) = \int_G f(g)\alpha_g(x)dg = x$ for each $x \in A^\alpha(K)$, we have for each $t \in G_i$

$$\alpha_t(x) = \alpha_t(\alpha_f(x)) = \int_G f(g)\alpha_t(\alpha_g(x))dg = \alpha_{f_t}(x)$$

where $f_t(s) = f(s-t)$. For each $\gamma \in U_K$ we have

$$\hat{f}_t(\gamma) = \int_G f(g-t)\gamma(g)dg = \int_G f(g)\gamma(g)\gamma(t)dg = \hat{f}(\gamma) = 1.$$

Hence we have $\alpha_t(x) = x$ for all $t \in G_i$ and $A^\alpha(K)$ is contained in A_i. Since $\cap G_i = \{e\}$, we have $\cup A(G_i) = \widehat{G}$. Therefore $\{A(G_i)|i \in I\}$ is the open covering of $\widehat{G}$. By the compactness of K there exists a finite subset $\{i_k\}_{k=1}^n$ of I such that $K \subset \cup A(G_{i_k})$. Hence there exists an index $i_0 \in I$ such that $K \subset A(G_{i_0})$. Since A_F^α is contained in $\cup A_i$, $\cup A_i$ is dense in A.

REMARK 3.6. If G is totally disconnected, then G has the property in Lemma 3.5.

Combining Theorem 3.4 and Lemma 3.5, we have the following.

COROLLARY 3.7. *Let (A, G, α) be a C^*-dynamical system with a compact abelian group G and G satisfy the same condition as in Lemma 3.5. Let δ be a densely defined closed *-derivation commuting with α on A. Assume that $(A_i, G/G_i, [\alpha]^i)$ be G/G_i-finite. Then δ is a generator of a C^*-dynamical system of A.*

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