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ON JOINT NUMERICAL RANGES AND JOINT SPECTRA OF LINEAR OPERATORS ON S.I.P. SPACES

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1. Introduction

G.Lumer [1] studied a vector space of type of inner product with a more general system of axiom than that of Hilbert space. He defined a semi-inner product on a vector space X as a complex (real) form $[x,y]$ on $X \times X$ which is linear in first one component only, strictly positive, and satisfies a Schwarz inequality. Such form induces a norm, by setting $[x,x]^{\frac{1}{2}}$, and for every normed linear space one can construct at least one such form (and in general, infinitely many) consistent with the norm in the sense $[x,x]^{\frac{1}{2}} = \|x\|$. In fact, every normed linear space can be made into a semi-inner-product space. In such a setting, one can then talk about a numerical range and spectrum of a bounded linear operator T on a semi-inner-product space X.

K.R.Unni and C.Puttamadaiah [2] studied a semi-inner-product and a bounded linear operator on a cartesian product of the two semi-inner-product space. Also, they showed that if A and B are bounded linear operators on homogeneous s.i.p. spaces X and Y respectively and $W(A)$ and $W(B)$ are convex subsets of complex numbers $\mathbf{C}$, then $W(A \oplus B) = C_0(W(A) \cup W(B))$.

In this paper if $S = (A_1, \ldots, A_n)$ and $T = (B_1, \ldots, B_n)$ are n-tupls of bounded linear operators on s.i.p. space X, then the joint numerical range $W(S \oplus T)$ is the convex hull of the union of $W(S)$ and $W(T)$. And joint spectrum $\sigma(S \oplus T)$ contains the union of $\sigma(S)$ and $\sigma(T)$.

2. Preliminaries and notations

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For completeness, we begin with the definition of s.i.p. space.

DEFINITION 2.1. *Let X be a complex vector space. A semi-inner-product on X is a complex function $[x,y]$ on $X \times X$ with the following properties:*
(1) $[\lambda x + y, z] = \lambda[x,z] + [y,z]$
(2) $[x,x] > 0$ for $x \neq 0$
(3) $|[x,y]|^2 \leq [x,x][y,y]$
for all x, y, z in X and for all complex numbers λ. A vector space with a semi-inner-product is called a semi-inner-product space (briefly s.i.p. space).

This definition has concrete significance by the following;

THEOREM 2.2.([1]). *A semi-inner-product space is a normed linear space with the norm*

$$\|x\| = [x,x]^{\frac{1}{2}}.$$

Conversely, every normed linear space can be made into a semi-inner-product space (in general, infinitely many different ways).

DEFINITION 2.3. *A s.i.p. space has homogeneity property when the s.i.p. satisfies*
(4) $[x, \lambda y] = \overline{\lambda}[x,y]$
for all x, y in X and for all complex numbers λ.

THEOREM 2.4.([3]). *Every normed linear space can be represented as a semi-inner-product with the homogeneity property.*

3. The joint numerical range and spectrum of bounded linear operator

If X and Y are s.i.p. space, then $X \oplus Y = \{(x,y) | x \in X,\ y \in Y\}$ is an s.i.p. space with componentwise addition, scalar multiplication together with the s.i.p. defined by

$$[(x_1, y_1), (x_2, y_2)] = [x_1, x_2] + [y_1, y_2].$$

The norm on $X \oplus Y$ is then given by

$$\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}.$$

If T_1 and T_2 are bounded linear operators on s.i.p. spaces X and Y respectively, then the bounded linear operator $T_1 \oplus T_2$ on $X \oplus Y$ is defined by

$$(T_1 \oplus T_2)(x,y) = (T_1 x, T_2 y).$$

DEFINITION 3.1. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded linear operators on s.i.p. space X. The joint numerical range $W(A)$ of A is the set of all points $Z = (Z_1, \ldots, Z_n)$ of $\mathbf{C}^n$ such that for some x in X, with $\|x\| = 1$, $Z_j = [A_j x, x]$ i.e.,*

$$W(A) = \{[Ax, x] = ([A_1 x, x], \ldots, [A_n x, x]) : \|x\| = 1\}.$$

THEOREM 3.2. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be n-tuples of bounded linear operators on homogeneous s.i.p. space X, Y respectively. If $W(A)$ and $W(B)$ are convex subsets of $\mathbf{C}^n$, then*

$$W(A \oplus B) = C_0(W(A) \cup W(B)),$$

where $C_0(S)$ denotes the convex hull of the set S.

Proof. Let $\lambda \in W(A \oplus B)$. We can find an element (x,y) in $X \oplus Y$ such that

$$\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}} = 1$$

and

$$\lambda = [(A \oplus B)(x,y), (x,y)] = [Ax, x] + [By, y].$$

Let $\|x\| = \alpha$, we see that $0 \leq \alpha \leq 1$ and $\|y\|^2 = 1 - \alpha$. Now $\lambda \in W(B)$ when $\alpha = 0$ and $\lambda \in W(A)$ for $\alpha = 1$. If $0 < \alpha < 1$, then

$$\lambda = \alpha[Ax', x'] + (1-\alpha)[By', y'],$$

where $x' = \dfrac{x}{\sqrt{\alpha}}$ and $y' = \dfrac{y}{\sqrt{1-\alpha}}$ are unit vectors in X and Y respectively. This shows that

$$\lambda \in C_0(W(A) \cup W(B)).$$

Conversely suppose $\lambda \in C_0(W(A) \cup W(B))$ so that

$$\lambda = \beta\mu + (1-\beta)\gamma$$

with $0 \leq \beta \leq 1$, $\mu \in W(A)$ and $\gamma \in W(B)$. There exist unit vectors x in X and y' in Y such that

$$\mu = [Ax, x] \text{ and } \gamma = [By, y].$$

Then

$$\begin{aligned}
\lambda &= \beta[Ax, x] + (1-\beta)[By, y] \\
&= [A\sqrt{\beta}x, \sqrt{\beta}x] + [B\sqrt{1-\beta}y, \sqrt{1-\beta}y] \\
&= [(A\sqrt{\beta}x, B\sqrt{1-\beta}y), (\sqrt{\beta}x, \sqrt{1-\beta}y)] \\
&= [(A \oplus B)(\sqrt{\beta}x, \sqrt{1-\beta}y), (\sqrt{\beta}x, \sqrt{1-\beta}y)]
\end{aligned}$$

Now

$$\begin{aligned}
\|(\sqrt{\beta}x, \sqrt{1-\beta}y\|^2 &= (\|\sqrt{\beta}x\|^2 + \|\sqrt{1-\beta}y\|^2) \\
&= \beta\|x\|^2 + (1-\beta)\|y\|^2 \\
&= \beta + (1-\beta) = 1.
\end{aligned}$$

Hence we conclude that $\lambda \in W(A \oplus B)$.

DEFINITION 3.3. *Let T be a bounded linear operator on a normed linear space X. If there exists $N > 0$ such that*

$$\|x\|N < \|Tx\| \text{ for all } x \text{ in } X.$$

We then call T bounded from below.

DEFINITION 3.4. *Let $A = (A_1, \dots, A_n)$ be an n-tuple of bounded linear operators on s.i.p. space X. The joint spectrum $\sigma(A)$ of A is the set of all points $\lambda = (\lambda_1, \dots, \lambda_n)$ of $\mathbf{C}^n$ such that $A_i - \lambda_i I$ is not invertible for each $i = 1, 2, \dots, n$ i.e.,*

$$\sigma(A) = \{(\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n | A_i - \lambda_i I \text{ is not invertible for each } i\},$$

where I denotes the identity operator on X.

LEMMA 3.4.([2]). *Let X and Y be s.i.p. spaces. Suppose $S : X \to X$ and $T : Y \to Y$ are bounded linear operators. Then;*

(a) *$S \oplus T$ is bounded from below if and only if S and T are both bounded from below.*

(b) *$\overline{R(S \oplus T)} = X \oplus Y$ if and only if $\overline{R(S)} = X$ and $\overline{R(T)} = Y$, where $\overline{R(U)}$ denote the closure of the range of the operator U.*

THEOREM 3.5. *If $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ are n-tuples of bounded linear operators on s.i.p. space X, Y respectively, then*

$$\sigma(A \oplus B) \supset \sigma(A) \cup \sigma(B).$$

Proof. Let I_X, I_Y and $I_{X \oplus Y}$ denote the identity operators on X, Y and $X \oplus Y$ respectively. Suppose $(\lambda_1, \ldots, \lambda_n) \notin \sigma(A \oplus B)$. Then for some i, $\lambda_i \notin \sigma(A_i \oplus B_i)$. Let $S_i = A_i - \lambda_i I_x$ and $T_i = B_i - \lambda_i I_y$. Then $S_i \oplus T_i = A_i \oplus B_i - \lambda_i I_{x \oplus y}$ and $S_i \oplus T_i$ is bounded from below, $\overline{R(S_i \oplus T_i)} = X \oplus Y$. By Lemma 3.4., S_i and T_i are bounded from below and $\overline{R(S_i)} = X$ and $\overline{R(T_i)} = Y$. This shows that $\lambda_i \notin \sigma(A_i)$ and $\lambda_i \notin \sigma(B_i)$ and hence $\lambda_i \notin \sigma(A_i) \cup \sigma(B_i)$. This completes the proof.

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