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ON INJECTIVITY PRESERVING MAPS AND INJECTIVE ELEMENTS ON $B(H)$ *

DONG-YUN SHIN+, SA-GE LEE+ AND CHANG HO BYUN++

0. Introduction

Many people has studied the theory of injective operator spaces([1], [2] e.t.c.).

In this paper we give examples of injectivity preserving maps on $B(H)$, study related operator spaces and injective operators. In section2, we show that on $B(H)$ $*$-operation and transpose map are injective but not 2-injective for dim $H \geq 2$. In section3, we define extremely injective space and show that for rank 1 projection p, $pB(H)$ is a maximal injective operator space and a C^*-algebra A is extremely injective if and only if dim $A \leq 2$. In section 4, we show that for a finite dimensional C^*-algebra A, an element $x \in A$ with $\|x\|=1$ is injective if and only if x is unitary, for a seperable Hilbert space H , an invertible element $x \in B(H)$ with $|x|=1$, x is injective if and only if x is unitary and for a C^*-algebra A, if $x \in A$ is an isometry, then x is left injective.

1. Preliminaries

We let M_n be the space of complex $n \times n$ matrices and $B(H)$ be the bounded operators on a Hilbert space H. M_n has the canonical basis $\{E_{ij}\}$ where E_{ij} is the matrix with 1 at (i,j) position and zero elsewhere. A linear subspace $E \subset B(H)$ is said to be an operator space.

Given operator space E, $M_n(E) = E \otimes M_n$ denotes the vector space of $n \times n$ matrices with entries in E.

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For $x = (x_{ij}) = \sum_{i,j}^{n} x_{ij} \otimes E_{ij} \in M_n(E)$ and $y = (y_{ij}) \in M_m(E)$, we write

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in M_{n+m}(E).$$

Identifying $M_n(B(H))$ with $B(H^n), M_n(E)$ can be regarded as an operator space of $B(H^n)$. Let $E \subset B(H)$ and $F \subset B(K)$ be operator spaces and $\phi : E \to F$ a linear or conjugate linear map. We define the map $\phi_n : M_n(E) \to M_n(F)$ by $\phi_n((x_{ij})) = (\phi(x_{ij}))$ for $(x_{ij}) \in M_n(E)$. We write, $\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in N\}$, where $\|\phi\| = \sup\{\|\phi(x)\| : x \in E, \|x\| = 1\}$. We call ϕ completely bounded if $\|\phi\|_{cb} < \infty$, and ϕ completely contractive if $\|\phi\|_{cb} \leq 1$.

Let E be an operator space. E is said to be an injective operator space if for every operator space F, every operator subspace F_0 of F and every completely bounded linear map $\phi : F_0 \to E$, there exists a linear map $\psi : F \to E$ such that $\psi|_{F_0} = \phi$ and $\|\psi\|_{cb} = \|\phi\|_{cb}$.

It is well known that $B(H)$ is an injective operator space for arbitrary Hilbert space H. A linear map $\phi : B(H) \to B(H)$ is called a completely contractive projection if $\|\phi\|_{cb} \leq 1$ and $\phi^2 = \phi$. Let $E \subset B(H)$ be an operator space. An E-projection of $B(H)$ is a completely contractive projection $\phi : B(H) \to B(H)$ such that $\phi(x) = x$ for all $x \in E$.

Let H be a Hilbert space and let $\{e_i\}_{i \in I}$ be a fixed orthonormal basis for H. For each $\xi = \sum a_i e_i \in H$ we set $\overline{\xi} = \sum \overline{a}_i e_i \in H$. For each $x \in B(H)$ and $\xi, \eta \in H$, we define $\theta(x)$ and $\tau(x)$ by $< \theta(x)\xi \mid \eta > = < x\overline{\eta} \mid \overline{\xi} >$ and $< \tau(x)\xi \mid \eta > = < \overline{\eta} \mid x\overline{\xi} >$, respectively. For each $x, y \in B(H)$, $\|x\| = \|\theta(x)\| = \|\tau(x)\|$, $\theta(xy) = \theta(y)\theta(x)$, $\theta(x)^* = \tau(x) = \theta(x^*)$, and $\tau(xy) = \tau(x)\tau(y)$. But $\theta(x)$ and $\tau(x)$ depend on orthonormal bases.

2. Examples of injectivity preserving maps on $B(H)$

THEOREM 2.1. *Let $E \subset B(H)$ be an operator space. Then E is injective if and only if there is an E-projection ϕ such that $\phi(B(H)) = E$.*

Proof. Combining Theorem 3.1. and Corollary 3.3([6]) completes the proof.

DEFINITION 2.2.. *Let $E \subset B(H)$ and $F \subset B(K)$ be operator spaces. A map $\phi : E \to F$ is an injectivity preserving map if $\phi(E_0)$ is injective* *whenever* E_0 *is an injective subspace of* E. *We say* ϕ *is n-injective if* ϕ_n *is an injectivity preserving map and completely injective if* ϕ *is n-injective for each positive inter n.*

THEOREM 2.3. *Let* $E \subset B(H)$ *and* $F \subset B(K)$ *be operator spaces and* $\phi : E \to F$ *be a completely contractive linear map which has completely contractive inverse. Then* ϕ *is completely injective.*

Proof. Let $N \subset M_n(E)$ be an injective opereator space. For every operator space M, every operator subspace L of M and every completely bounded linear map $\psi : L \to \phi_n(N), {\phi_n}^{-1} \circ \psi : L \to N$ is completely bounded. Hence there is a linear map $\tau : M \to N$ such that $\tau|_L = {\phi_n}^{-1} \circ \psi$ and $\|\tau\|_{cb} = \|{\phi_n}^{-1} \circ \psi\|_{cb} = \|\psi\|_{cb}$. Then the linear map $\phi_n \circ \tau : M \to \phi_n(N)$ is an extension of ψ such that $\|\phi_n \circ \tau\|_{cb} = \|\psi\|_{cb}$.

COROLLARY 2.4. *Let* $E \subset B(H)$ *and* $F \subset B(K)$ *be operator spaces and* $\phi : E \to F$ *be a linear bijection with* $\|\phi\|_{cb}\|\phi^{-1}\|_{cb} = 1$.*Then* ϕ *is completely injective.*

Proof. Put $\psi = \frac{\phi}{\|\phi\|_{cb}}$. Then $\psi^{-1} = \frac{\phi^{-1}}{\|\phi^{-1}\|_{cb}}$. By Theorem2. 3, ψ is completely injective. Hence ψ is completely injective.

COROLLARY 2.5. *Let* $A \subset B(H)$ *be a* C^**-algebra and* $\phi : A \to B(K)$ *be a* $*$*-isomorphism. Then* ϕ *is completely injective.*

COROLLARY 2.6. *Let* ϕ *be a* $*-$ *automorphism on* $B(H)$. *Then* ϕ *is completely injective.*

LEMMA 2.7. *Let* $\alpha : B(H) \to B(H)$ *be a bijection with* $\alpha \circ \alpha = id, \phi$ *be an E-projection with* $\phi(B(H)) = E$ *and* $\psi = \alpha \circ \phi \circ \alpha$. *Then* $\psi \circ \psi = \psi, \psi|_{\alpha(E)} = id_{\alpha(E)}$ *and* $\psi(B(H)) = \alpha(E)$.

Proof. It is an easy compuation.

THEOREM 2.8. *Let* $* : B(H) \to B(H)$ *be the map defined by* $*(x) = x^*$ *Then* $*$ *is an injectivity preserving map.*

Proof. Let $E \subset B(H)$ be an injective operator space. By Theorem 2. 1, there is an E-projection ϕ with $\phi(B(H)) = E$. We denote $*(E) = E^*$.

Let $\phi^* = * \circ \phi \circ *$. Then by Lemma 2. 7, $\phi^* \circ \phi^* = \phi^*, \phi^*(x) = x$ for all $x \in E^*$ and $\phi^*(B(H)) = E^*$. Hence to complete the proof, we must show that ϕ^* is completely contractive. Let $\sum_{i,j=1}^n x_{ij} \otimes E_{ij} \in M_n(B(H))$ and $\xi = (\xi_1, \cdots, \xi_n)^t$, $\eta = (\eta_1, \cdots, \eta_n)^t \in H^n$. Then $< \sum_{i,j}^n x_{ij}^* \otimes E_{ij}\xi \mid \eta > = < \xi \mid \sum_{i,j=1}^n x_{ij} \otimes E_{ji}\eta > = \overline{< \sum_{i,j=1}^n x_{ij} \otimes E_{ji}\eta \mid \xi >}$. Hence $\| \sum_{i,j=1}^n x_{ij}^* \otimes E_{ij}\| = \| \sum_{i,j=1}^n x_{ji} \otimes E_{ij}\|$. Therefore

$$\|\phi_n^*(\sum_{i,j=1}^n x_{ij} \otimes E_{ij})\| = \| \sum_{i,j=1}^n \phi^*(x_{ij}) \otimes E_{ij}\|$$

$$= \| \sum_{i,j}^n \phi(x_{ij}^*) \otimes E_{ji}\| = \|\phi_n(\sum_{i,j=1}^n x_{ij}^* \otimes E_{ji})\|$$

$$\leq \| \sum_{i,j=1}^n x_{ij}^* \otimes E_{ji}\| = \| \sum_{i,j=1}^n x_{ij} \otimes E_{ij}\|.$$

Hence ϕ^* is completely contractive.

REMARK 2.9. *$* : M_2 \to M_2$ is not 2-injective since $*_2(aE_{11} + bE_{12} + cE_{14}) = \overline{a}E_{11} + \overline{b}E_{21} + \overline{c}E_{23}$. Hence $* : B(H) \to B(H)$ is not 2-injective whenever dim $H \geq 2$.*

REMARK 2.10. *Let A be a C^*-algebra. Since A can be embeded in $B(H)$ for some Hilbert space H, the map $*$ on A is an injectivity preserving map.*

COROLLARY 2.11. *Let $\phi : B(H) \to B(H)$ be a conjugate linear, $*$-preserving bijection with $\phi(xy) = \phi(y)\phi(x)$. Then ϕ is an injectivity preserving map.*

Proof. Since $* \circ \phi$ is a $*$-automorphism on $B(H), \phi = * \circ (* \circ \phi)$ is an injectivity preserving map.

THEOREM 2.12. *Let $\{e_i\}$ be an orthonormal basis for a Hilbert space H and θ the transpose map with respect to this basis. Then θ is an injectivity preserving map.*

Proof. Let $E \subset B(H)$ be an injective operator space. By Theorem 2. 1, there is an E-projection ϕ with $\phi(B(H)) = E$. Define $\phi^t(x) : B(H) \to$

$B(H)$ by $\phi^t(x) = \theta(\phi(\theta(x)))$. Then by Lemma 2. 7, $\phi^t \circ \phi^t = \phi^t, \phi^t(x) = x$ for $x \in \theta(E)$ and $\phi^t(B(H)) = \theta(E)$. Hence to complete the proof, we must show that ϕ^t is completely contractive. Let $\sum_{i,j=1}^n x_{ij} \otimes E_{ij} \in M_n(B(H))$ and $\xi = (\xi_1, \cdots, \xi_n)^t$, $\eta = \eta_1, \cdots, \eta_n)^t \in H^n$. Then

$$< \sum_{i,j=1}^n \theta(x_{ij}) \otimes E_{ij}\xi \mid \eta >= \sum_{i,j=1}^n < \theta(x_{ij})\xi_j \mid \eta_i >$$
$$= \sum_{i,j=1}^n < x_{ij}\overline{\eta_i} \mid \overline{\xi_j} >=< \sum_{i,j=1}^n x_{ij} \otimes E_{ji}\overline{\eta} \mid \overline{\xi} >,$$

where $\overline{\eta} = (\overline{\eta}_1, \cdots, \overline{\eta}_n)^t$ and $\overline{\xi} = (\overline{\xi}_1, \cdots, \overline{\xi}_n)^t$. Hence $\| \sum_{i,j=1}^n \theta(x_{ij}) \otimes E_{ij}\| = \| \sum_{i,j=1}^n x_{ij} \otimes E_{ji}\|$. Therefore

$$\|{\phi^t}_n(\sum_{i,j=1}^n x_{ij} \otimes E_{ij})\| = \| \sum_{i,j=1}^n \phi^t(x_{ij}) \otimes E_{ij}\|$$
$$=\| \sum_{i,j=1}^n \phi(\theta(x_{ji})) \otimes E_{ij}\| = \|\phi_n(\sum_{i,j=1}^n \theta(x_{ji}) \otimes E_{ij})\|$$
$$\leq\| \sum_{i,j=1}^n \theta(x_{ji} \otimes E_{ij})\| = \| \sum_{i,j=1}^n x_{ij} \otimes E_{ij}\|.$$

COROLLARY 2.13. *Let $\phi : B(H) \to B(H)$ be a linear $*$-preserving bijection with $\phi(xy) = \phi(y)\phi(x)$.Then ϕ is an injectivity preserving map.*

Proof. The same as the proof of Corollaey 2.11.

COROLLARY 2.14. *Let $\{e_i\}_{i\in I}$ be an orthonormal basis for a Hilbert space H. Let τ be a map defined by $< \tau(x)\xi \mid \eta >=< \overline{\eta} \mid \overline{\xi} >$. Then τ is completely injective.*

Proof. By elementary caculation, $\tau = * \circ \theta (= \theta \circ *)$. Hence τ is an injectivity preserving map. It is easy to show that the map $\tau_n : M_n(B(H)) \to M_n(B(H))$ is the map $\tau : B(H \otimes C^n) \to B(H \otimes C^n)$ with basis $\{e_i \otimes E_k : i \in I, 1 \leq k \leq n\}$. Hence τ is completely injective.

Let $H = C^2$ with basis $\{(1,0),(0,1)\}$. Then $\theta : M_2 \to M_2$ is not 2-injective since

$$\theta_2(aE_{11} + bE_{12} + cE_{14}) = aE_{11} + bE_{21} + cE_{23}.$$

Let H be a Hilbert space and $\alpha = \{e_i\}_{i \in I}$ and $\beta = \{f_i\}_{i \in I}$ be two orthonormal basis for H. Let U be the unitary operator with $Ue_i = f_i$ and $\theta_\alpha, \theta_\beta$ be the transpose maps with respect to the bases α and β. Then

$$\begin{aligned} < \theta_\alpha(x)e_i \mid e_j > &=< xe_j \mid e_i > \\ &=< xU^*f_j \mid U^*f_i > \\ &=< UxU^*f_j \mid f_i > \\ &=< \theta_\beta(UxU^*)f_i \mid f_j > \\ &=< \theta_\beta(UxU^*)Ue_i \mid Ue_j > \\ &=< U^*\theta_\beta(UxU^*)Ue_i \mid e_j > . \end{aligned}$$

Hence $\theta_\alpha(x) = U^*\theta_\beta(UxU^*)U$. Therefore θ_α is n-injective if and only if $\theta_\beta$is n-injective. Since $\theta : M_2 \to M_2$ is not 2-injective, $\theta : B(H) \to B(H)$ is not 2-injective whenever dim $H \geq 2$.

If dim $H = 1, \theta = id$. Hence θ is completely injective. Since $\theta = * \circ \tau, *$ is 2-injective if and only if dim $H \leq 1$. Hence we have shown the following theorem.

THEOREM 2.15. *Let H be a Hilbert space with basis $\alpha = \{e_i\}$. Then the following are equivalent:*

(1) *The map $* : B(H) \to B(H)$ is 2-injective.*
(2) *The map $* : B(H) \to B(H)$ is completely injective.*
(3) *dim $H \leq 1$.*
(4) *The transpose map θ_α is 2-injective.*
(5) *The transpose map θ_α is completely injective.*

COROLLARY 2.16. *Let $\phi : B(H) \to B(H)$ be a conjugate linear, $*$-preserving bijection with $\phi(xy) = \phi(y)\phi(x)$ and $\psi : B(H) \to B(H)$ be a linear, $*$-preserving bijection with $\psi(xy) = \psi(y)\psi(x)$. Then the following are equivalent:*

(1) *ϕ is 2-injective.*

(2) ϕ *is completely injective.*
(3) *dim* $H \leq 1$.
(4) ψ *is 2-injective.*
(5) ψ *is completely injective.*

THEOREM 2.17. *Let* $E \subset B(H), F \subset B(K)$ *be operator spaces and* $\phi : E \to F$ *be (n+1)-injective. Then* ϕ *is n-injective.*

Proof. Let L be an injective operator space contained in $M_n(E)$. We denote $L \oplus 0 = \{x \oplus 0 : x \in L, 0 \in B(H)\} \subset M_{n+1}(E)$. Then $L \oplus 0$ is injective. Since $\phi_{n+1}(L \oplus 0) = \phi_n(L) \oplus 0$ is injective, $\phi_n(L)$ is injective.

3. Extremely injective spaces

DEFINITION 3.1.. *An operator space* E *is called (finitely) extremely injective if its (finite dimensional) closed subspaces are injective.*

THEOREM 3.2. *Let* $p \in B(H)$ *be a rank 1 projection in* $B(H)$. *Then* $B(H)p$ *is extremely injective.*

Proof. Choose a unit vector η in the range of p. For each $x, y \in B(H)$, define $\phi : B(H)p \to H$ by $\phi(xp) = x\eta$ and $< xp \mid yp >=< x\eta \mid y\eta >$, where $< x\eta \mid y\eta >$ is the inner product in H. Then $(B(H)p, < \mid >)$ is a Hilbert space and ϕ is an isometric isomorphism. Let E be a closed subspace of $B(H)p$. Then $\phi(E)$ is a closed subspace of H. Hence there is the projection $q \in B(H)$ with $\phi(E) = qH$. Therefore $E = qB(H)p$ and E is injective.

COROLLARY 3.3. *Let* $p \in B(H)$ *be a rank 1 projection. Then* $pB(H)$ *is extremely injective.*

Proof. Since $pB(H) = (B(H)p)^*$ and $*$ is an injectivity preserving map, $pB(H)$ is extremely injective.

LEMMA 3.4. *Let* $x = (x_{ij}) \in M_n$ *with* $x_{1i} = 0$ $(1 \leq i \leq n)$, E=*Span*$\{E_{11}M_n, x\}$ *and* $\phi : M_n \to E$ *be an* E*-projection. Then for* $i \geq 2$, $\phi(E_{ij}) = b_{ij}x$ *for some* $b_{ij} \in C$.

Proof. For $i \geq 2$, put $\phi(E_{ij}) = \sum_{l=1}^{n} a_l E_{1l} + b_{ij}x$ for some $a_l, b_{ij} \in C$. Since $\phi(E_{1k}) = E_{1k}, \phi(E_{ij} + mE_{1k}) = \sum_{l=1}^{n} b_l E_{1l} + b_{ij}x$, where $b_l = a_l$

for $l \neq k$ and $b_k = a_k + m$. By elementry caculation, $\|E_{ij} + mE_{1k}\| \leq \sqrt{1+m^2}$ and $\|\phi(E_{ij} + mE_{1k})\| = \|\sum_{l=1}^{n} b_l E_{1l} + b_{ij}x\| \geq \mid b_k \mid = \mid a_k + m \mid$. This implies $\sqrt{1+m^2} \geq \mid a_k + m \mid$ for each $m \in C$. Hence $a_k = 0$ and $\phi(E_{ij}) = b_{ij}x$.

THEOREM 3.5. *Let $E_{11}M_n \subset E \subset M_n$ and $\dim E = n+1$. Then E is not injective.*

Proof. We can choose $x = (x_{ij}) \in E$ with $\|x\| = 1$ and $x_{1i} = 0$ for $1 \leq i \leq n$. Suppose E is injective. Then there is an E-projection $\phi : M_n \to E$.

Case 1.

There exist $i, j (i \neq j)$ such that $xE_{ii} \neq 0, xE_{jj} \neq 0$. By Lemma 3.4, $\phi(E_{kl}) = b_{kl}$ for $2 \leq k \leq n, 1 \leq l \leq n$. For $l \neq i$, $\|E_{kl} + E_{1i}\| = 1$ and $\|\phi(E_{kl} + E_{1i})\| = \|E_{1i} + b_{kl}x\| \geq \|E_{1i} + b_{kl}xE_{ii}\| = \sqrt{1 + \mid b_{kl}|^2 \|xE_{ii}\|^2}$. Hence $b_{kl} = 0$. By the same way, $b_{kl} = 0$ for $l \neq j$. Hence $b_{kl} = 0$ for $2 \leq k \leq n, 1 \leq l \leq n$ and $\phi(x) = 0$. It is a contradiction.

Case 2. There is only one i such that $xE_{ii} \neq 0$. We may assume $i = 1$ and $x_{21} \neq 0$. By Lemma 3.4, $\phi(E_{22}) = b_{22}x$. Since $\phi(E_{11} + E_{22}) = E_{11} + b_{22}x$, $\|E_{11} + E_{22}\| = 1$ and $\|E_{11} + b_{22}x\| = \sqrt{1 + |b_{22}|^2}$, $b_{22} = 0$. Hence $\phi(E_{22}) = 0$. We have

$$\begin{aligned}
&\|E_{11} + x_{21}E_{12} - E_{22} + x\|^2 \\
=&\|(E_{11} + x_{21}E_{12} - E_{22} + x)^*(E_{11} + x_{21}E_{12} - E_{22} + x)\| \\
=&\| \begin{pmatrix} 2 & 0 \\ 0 & 1 + |x_{21}|^2 \end{pmatrix} \| = 2.
\end{aligned}$$

Since $\phi(E_{11} + x_{21}E_{12} - E_{22} + x) = E_{11} + x_{21}E_{12} + x$,

$$\begin{aligned}
&\|\phi(E_{11} + x_{21}E_{12} - E_{22} + x)\|^2 \\
=&\|(E_{11} + x_{21}E_{12} + x)^*(E_{11} + x_{21}E_{12} + x)\| \\
=&\| \begin{pmatrix} 2 & x_{21} \\ x_{21} & |x_{21}|^2 \end{pmatrix} \| > 2 + \frac{1}{2}|x_{21}|^2.
\end{aligned}$$

Hence ϕ is not contractive and it is a contradiction. Therefore E is not injective.

Theorem 3.5 implies that $E_{11}M_n$ is a maximal extremely injective operator subspace of M_n.

COROLLARY 3.6. *Let $p \in B(H)$ be a rank 1 projection. Then $pB(H)$ is a maximal extremely injective operator subspace of $B(H)$.*

PROPOSITION 3.7. *Let $\sum_{l=1}^{n} |a_{kl}| \leq 1$ for $1 \leq k \leq m$ and $E = \{\sum_{k=1}^{n} b_k \ (E_{kk} + \sum_{l=1}^{m} a_{kl}E_{n+ln+l}) : b_1, \ldots, b_n \in C\}$. Then E is injective.*

Proof. Define $\phi : M_{n+m} \rightarrow E(C^{n+m} \subset M_{n+m})$ with $\phi(E_{kk}) = E_{kk} + \sum_{k=1}^{m} a_{kl}E_{n+kn+k}$ for $1 \leq k \leq n$, and $\phi(E_{kl}) = 0$ for otherwise. Then $\phi \circ \phi = \phi$ and $\phi|_E = id$, and $\phi(B) = \sum_{k=1}^{n} b_{kk}\phi(E_{kk})$ for an $(n+m)$ matrix $B = (b_{ij})$. Hence $\|\phi(B)\|$=max $\{|b_{kk}| : 1 \leq k \leq n\} \leq \|B\|, \|\phi\| = 1$. Since $E \subset C^{n+m}$, $\|\phi\|_{cb} = \|\phi\| = 1$([4], Theorem3. 8.). Therefore E is injective.

PROPOSITION 3.8. *Let $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ be fixed and $E = \{\sum_{k=1}^{n} b_k(E_{kk} + a_kE_{n+1n+1}) : b_1, b_2, \ldots, b_n \in C\}$. Then E is injective if and only if $\sum_{k=1}^{n} a_k \leq 1$ or $1 + a_1 + \cdots + a_{n-1} \leq a_n$.*

Proof. ($\Leftarrow$) Case 1. $\sum_{k=1}^{n} a_k \leq 1$. By Proposition 3.7, E is injective.

Case 2 . $1 + a_1 + \cdots + a_{n-1} \leq a_n$. By elementary caculation, $E = \{b_n(\frac{1}{a_n}E_{nn} + E_{n+1n+1}) + \sum_{k=1}^{n-1} b_k(E_{kk} - \frac{a_k}{a_n}E_{nn}) : b_1, ..., b_n \in C\}$, and E is injective.

($\Rightarrow$) Let E be injective and $1 + a_1 + \cdots + a_{n-1} > a_n$. Since E is injective, there is an E-projection $\phi : M_{n+1} \rightarrow M_{n+1}$ with $\phi(M_{n+1}) = E$. Hence there are complex numbers c_{ij} for $1 \leq i,j \leq n+1$ such that $\phi(E_{kk}) = \sum_{i=1}^{n} c_{ki}(E_{ii} + a_iE_{n+1n+1})$ for $1 \leq k \leq n+1$. Since $\phi(E_{kk} + a_kE_{n+1n+1}) = E_{kk} + a_kE_{n+1n+1}$ for $1 \leq k \leq n$, $\phi(E_{n+1n+1}) = \frac{1}{a_k}(E_{kk} + a_kE_{n+1n+1} - \sum_{i=1}^{n} c_{ki}(E_{ii} + a_iE_{n+1n+1}))$ for $1 \leq k \leq n$, and $c_{n+1k} = \frac{1-c_{kk}}{a_k} = \frac{-c_{lk}}{a_l}$ for $1 \leq l, k (l \neq k) \leq n$. Since $|c_{kk}| \leq 1$ and $a_kc_{n+1k} = 1 - c_{kk}$, Re $c_{n+1k} \geq 0$. Since $E_{kk}\phi(2E_{kk} + 2E_{n+1n+1} - I) = (2c_{kk} + 2c_{n+1k} - \sum_{i=1}^{n+1} c_{ik})E_{kk} = \{1 + (1 - 2a_k + \sum_{i=1}^{n} a_i)c_{n+1k}\}E_{kk}$, Re $(1 - 2a_k + \sum_{i=1}^{n} a_i)c_{n+1k} \leq 0$ for $1 \leq k \leq n$. Since $1 - 2a_k + \sum_{i=1}^{n} a_i > 0$ for $1 \leq k \leq n$, Re $c_{n+1k} = 0$ for $1 \leq k \leq n$. Hence $c_{kk} = 1$ for $1 \leq k \leq n$ and $c_{kl} = 0$ for otherwise. Then $\phi(I)E_{n+1n+1} = \sum_{k=1}^{n} a_kE_{n+1n+1}$. Therefore $\sum_{k=1}^{n} a_k \leq 1$.

Proposition 3.8 implies that for a positive inter n C^*-algebra C^n is extremely injective if and only if $n \leq 2$.

COROLLARY 3.9. *Let A be a $C*$-algebra. Then A is extremely injective if and only if dim $A \leq 2$.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Case 1. $3 \leq \dim A < \infty$.
Since dim $A < \infty, A$ is decomposed into the direct sum $A = \oplus_{k=1}^n A_k$, where each A_k is isomorphic to the algebra of $n_k \times n_k$-matrices ([7], Theorem 1.11.2.) Hence A is not extremely injective.

Case 2. dim A is infinite.
Since A is infinite dimensional C^*-algebra, there is a positive element x with infinite spectrum ([3], Exersise 6.14.). Choose $\lambda_1, \lambda_2, \lambda_3 \in Sp(x)$ with $0 < \lambda_1 < \lambda_2 < \lambda_3$. Put $\lambda_0 = 0$ and $\lambda_4 = 1 + \lambda_3$. Define $f_i(i = 1,2,3,) : [0, \infty) \to [0,1]$ with

$$f_i(\lambda) = \begin{cases} \dfrac{2\lambda - \lambda_{i-1} - \lambda_i}{\lambda_i - \lambda_{i-1}} \text{ for } \lambda_{i-1} + \lambda_i \leq 2\lambda \leq 2\lambda_i, \\ \dfrac{2\lambda - \lambda_i - \lambda_{i+1}}{\lambda_i - \lambda_{i+1}} \text{ for } 2\lambda_i \leq 2\lambda \leq \lambda_i + \lambda_{i+1}, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

Then $f_i(x) \in A, f_i(x)f_j(x) = 0$ for $i \neq j$ and $\|af_1(x) + bf_2(x) + cf_3(x)\|$=max $\{|a|, |b|, |c|\}$. Hence, by the same way in the proof of Proposition 3.8, $E = \{af_1(x) + bf_2(x) - (a+b)f_3(x) : a, b \in C\} \subset A$ is not injective. Therefore A is not extremely injective.

THEOREM 3.10. *Let $E \subset B(H)$ be an operator space such that dim E is at most countable. Then the following are equivalent:*

(1) *E is extremely injective.*
(2) *E is injective and for each operator space F and any linear map $\phi : F \to E, \phi$ is an injectivity preserving map.*
(3) *E is injective and for each operator space F of E, and any linear map $\phi : F \to E, \phi$ is an injectivity preserving map.*
(4) *E is injective and for any linear map $\phi : E \to E$ is an injectivity preserving map.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Clear.

(4) $\Rightarrow$ (1). Let $F \subset E$ be a subspace. Choose a basis $\{x_i\}_{i\in I}$ of F and a basis $\{x_j\}_{j\in J}$ of E with $I \subset J$. Define a linear map $\phi : E \to E$ by $\phi(x_i) = x_i$ for $i \in I$ and $\phi(x_i) = 0$ for $i \in J \setminus I$. Hence $F = \phi(E)$ is injective.

For operator spaces E and F, the set of all injectivity preserving linear maps $\phi : E \to F$ will be denoted by $IP(E,F)$. And $\#IP(E,F)$ denotes the supremum of all dimensions of subspaces of $IP(E,F)$. We set $IP(E) = IP(E,E)$. In general, $IP(E,F)$ is not a vector space. If F is extremely injective or dim $E \leq 1$, then $IP(E,F)$ is a vector space but the converse is not known. For an operator space E, $I(E)$ denotes the set of all extremely injective subspace of E. And $\#I(E)$ denotes the supremum of all dimensions of subspaces of $I(E)$.

Let E and F be finite dimensional operator spaces, let $F_0 \subset F$ be an extremely injective subspace, let $\{e_1, e_2, ..., e_n\}$ be a basis for E, and let $\{f_1, f_2, ..., f_k\}$ be a basis for F_0. For $1 \leq i \leq n, 1 \leq j \leq k$, define $\phi_{ij} : E \to F$ by $\phi_{ij}(e_l) = \delta_{il} f_j$. Then $\{\phi_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ is linearly independent and Span$\{\phi_{ij} : 1 \leq i \leq n, 1 \leq j] \leq k\} \subset IP(E,F)$. Hence dim $E \cdot \#I(F) \leq \#IP(E,F) \leq$ dim $E\cdot$ dim F. In particular, $\#IP(E,F) =$ dim $E\cdot$ dim F whenever F is extremely injective. Since $E_{11}M_n$ is extremely injective, $\#I(M_n) \geq n$ and $\#IP(M_n) \geq n^3$.

4. Injective elements in $C*$-algebras

For a C^*-algebra A and $x, y \in A$, let L_x and R_x be a linear map defined by $L_x y = xy$ and $R_x y = yx$.

DEFINITION 4.1.. *For a C^*-algebra A, an element $x \in A$ is called left (resp. right) injective if L_x(resp. R_x) is an injectivity preserving map. An element $x \in A$ is injective if x is left and right injective.*

Obviously a unitary element $x \in A$ is injective. Since $L_x E = (R_{x^*} E^*)^*$ and $*$-operation is an injectivity preserving map, x is left injective if and only if x^* is right injective.

LEMMA 4.2. *Let $x = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \in M_2$ and $ab \neq 0$ Then x is not left* *injective.*

Proof. Put $E = \{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in \mathbf{C}\}$. Then E is injective. Suppose x is left injective. Then

$$L_x E = \text{Span}\{\begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\} \text{ is injective.}$$

Hence there is an L_xE-projection $\phi : M_2 \to L_xE$. Put $\phi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \alpha\begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since $\phi\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \alpha a & \beta \pm 1 \end{pmatrix}$ and $\|\begin{pmatrix} \alpha & 0 \\ \alpha a & \beta \pm 1 \end{pmatrix}\| \leq 1, \alpha = \beta = 0$. Since $\phi\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = \phi\begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} - \phi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\phi\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}$, a contradiction. Therefore x is not injective.

LEMMA 4.3. *Let $x = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \in M_2$ and $ab \neq 0$. Then x is not left injective.*

Proof. By the same method in the proof of Lemma 4.2, it is trivial.

LEMMA 4.4. *Let $x = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \in M_2$ and $|a| > 1$. Then x is not left injective.*

Proof. Put $E = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in C\}$. Then E is injective. Suppose x is left injective. Then L_xE=Span $\{\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}\}$ is injective. Hence there is an L_xE-projection $\phi : M_2 \to L_xE$. Put $\phi(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \alpha\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$. Since $\phi(\begin{pmatrix} k+1 & 0 \\ 0 & ka \end{pmatrix}) = k\phi(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix})+ \phi(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = (k+\alpha)\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, |ka| \geq |(k+\alpha)a|$ for sufficiently large $k \in \mathbf{C}$. Hence $\alpha = 0$. Since $\phi\begin{pmatrix} 1 & k \\ ka & 0 \end{pmatrix} = (\beta + k)\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$ for all $k \in \mathbf{C}$, $\beta = 0$. Hence $\phi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Similarly $\phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $\phi\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ and $\phi\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$. Then $\phi\begin{pmatrix} 0 & 0 \\ a & a \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ a & a \end{pmatrix}$. Since $\| \begin{pmatrix} 0 & 0 \\ a & a \end{pmatrix} \| = \sqrt{2}|a|$ and $\| \begin{pmatrix} 1 & 1 \\ a & a \end{pmatrix} \| = \sqrt{2(1+|a|^2}$, ϕ is not contractive. It is a contradiction. Therefore x is not left injective.

Corollary 4.5. *Let* $x = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \in M_2$ *and* $0 < |a| < 1$. *Then* x *is not left injective.*

Proof. Since $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, x is not left injective.

Corollary 4.6. *Let* $x = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$ *or* $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \in M_2$ *and* $b \neq 0$. *Then* x *is left injective if and only if* $a = 0$ *and* $|b| = 1$.

Proof. ($\Leftarrow$) Since $a = 0$, $|b| = 1$, x is unitary and x is injective.

($\Rightarrow$) By Lemma 4.2 and Lemma 4.3, $a = 0$. By Lemma 4.4 and Corollary 4.5, $|b| = 1$.

Lemma 4.7. $E = \{\begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in \mathbf{C}\}$ *is not injective.*

Proof. Suppose E is injective. Then there is an E-projection $\phi : M_2 \to M_2$ with $\phi(M_2) = E$. Put $\phi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$. Since $\phi\begin{pmatrix} 1 & 0 \\ k & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c+k & a \end{pmatrix}$ and $|c+k| \leq \sqrt{1+|k|^2}$ for all $k \in \mathbf{C}$, $c = 0$. Similarly $b = 0$ and $\phi\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ with $a + d = 1$. Since ϕ is unital contraction, ϕ is completely positive ([4], Proposition 2. 11). Since ϕ is completely positive if and only if $\begin{pmatrix} \phi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \phi\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \phi\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$ is positive ([4], Theorem 3. 12). Thus $\begin{pmatrix} a & 0 & 0 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1-a & 0 \\ 1 & 0 & 0 & 1-a \end{pmatrix}$ is positive. But

$$< \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1-a & 0 \\ 1 & 0 & 0 & 1-a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \mid \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} >= -1$$

Therefore E is not injective.

LEMMA 4.8. *Let* $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_3$. *Then x is not left injective.*

Proof. Put $E = \{ \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} : a, b, c \in \mathbf{C}\}$. Since E is a commutative C^*-algebra with $\dim E = 3$, E is injective. By Lemma 4.7, xEx is not injective. Hence x is not left injective.

LEMMA 4.9. *Let $x = \sum_{i=1}^{n} \lambda_i E_{ii} \in M_n$ with $\lambda_1 = 1$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Then x is left injective if and only if $\lambda_i = 1$ for $1 \leq i \leq n$ or $\lambda_i = 0$ for $2 \leq i \leq n$.*

Proof. ($\Leftarrow$) Since $x = I$ or x is a projection of rank 1, x is injective.
($\Rightarrow$) Suppose $\lambda_2 \neq 0$. Since x is left injective, $\begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ is left injective. By Corollary 4.6 $\lambda_2 = 1$ and $\lambda_3 = 1$ or 0. By Lemma 4.8, $\lambda_3 = 1$. Simiarly $\lambda_k = 1$ for $1 \leq k \leq n$.

COROLLARY 4.10. *Let $x \in B(H)$ be a non-zero projection. Then x is injective if and only if $x = I$ or rank $x = 1$.*

THEOREM 4.11. *Let $x \in M_n$ with $\|x\| = 1$. Then the following are equivalent:*

(1) *x is injective*
(2) *x is left injective*
(3) *x is right injective*
(4) *x is unitary or rank of x is 1.*

Proof. (1)$\Rightarrow$(2) trivial.

(2)$\Rightarrow$(4) Since $x \in M_n$ and $\|x\| = 1$, there are unitary matrices U and V, diagonal matrix $D = \sum_{k=1}^{n} \lambda_k E_{kk}$ with $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ with $x = UVDV^*$. Since x is left injective, D is left injective. Hence by Lemma 4.9, $D = I$ or $D = E_{11}$. Therefore x is unitary or rank $x = 1$.

(4)$\Rightarrow$(1) For rank $x = 1$, there are unitary matrices U and V such that $x = UE_{11}V$. Hence if rank $x = 1$, x is injective.

(3) $\Leftrightarrow$ (4) Since x is left injective if and only if x^* is right injective and rank x = rank x^*, it is obvious.

LEMMA 4.12. *Let H be a separable Hilbert space and $x \in B(H)$ be invertible, $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Then there is an invertable operator $y \in B(H)$ such that xy is unitary in $B(H)$, $< ye_k | e_n >= 0$ for $k < n$ and $< ye_n \mid e_n >> 0$.*

Proof. Since x is invertible, $\{xe_n\}_{n=1}^{\infty}$ forms a basis for H. Let $xe_n = \beta_n$ and $\alpha_1, \cdots, \alpha_n$ be the vectors obtained by the Gram-Schmidt process. Then for each $n \in N$, $\{\alpha_1, \cdots, \alpha_n\}$ is an orthonormal basis for the subspace spanned by $\{\beta_1, \cdots, \beta_n\}$ and

$$\alpha_n = \beta_n - \sum_{k=1}^{n-1} \frac{< \beta_n \mid \alpha_k >}{\|\alpha_k\|^2} \alpha_k.$$

Hence, for each n there exist unique scalars c_{nk} such that $\alpha_n = \beta_n - \sum_{k=1}^{n-1} c_{kn}\beta_k$. Let U be the unitary operator with $U(e_n) = \frac{\alpha_n}{\|\alpha_n\|'}$, and y be the operator defined by

$$y(e_n) = \frac{1}{\|\alpha_n\|} e_n - \frac{1}{\|\alpha_n\|}(c_{1n}e_1 + \cdots + c_{n-1n}e_{n-1}).$$

Then $xy(e_n) = \frac{1}{\|\alpha_n\|}\beta_n - \frac{1}{\|\alpha_n\|}(c_{1n}\beta_1 + \cdots + c_{n-1n}\beta_{n-1}) = \frac{\alpha_n}{\|\alpha_n\|}$. Hence $U = xy, x^{-1}U = y \in B(H), < ye_k \mid e_n) = 0$ for $k < n$ and $< ye_n \mid e_n >= \frac{1}{\|\alpha_n\|} > 0$.

LEMMA 4.13. *Let H be a separable Hilbert space and $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Let $x \in B(H)$ be invertible with $< xe_k \mid e_n >= 0$ for $k < n$, $< xe_n|e_n >> 0$, $\|x\| = 1$ and x is left injective. Then $x = I$.*

Proof. Put p_k be the projection with Ran $p_k =< e_k >$, and $E_k = \{ap_1 + bp_k : a, b \in \mathbf{C}\}$ for $k > 1$. Then E_k is injective and xE_k is injective. By Corollary 4.6, $< xe_1|e_1 >=< xe_k|e_k >$ and $< xe_1|e_k >= 0$ for $1 \neq k$. Similary $< xe_n|e_k >= 0$ for $k \neq n$. Thus $x =< xe_1|e_1 > I = I$.

THEOREM 4.14. *Let H be a separable Hilbert space, and $x \in B(H)$ be invertible with $\|x\| = 1$. Then the following are equivalent:*

(1) *x is injective*
(2) *x is left injective*
(3) *x is right injective*
(4) *x is unitary.*

Proof. (1)$\Rightarrow$(2) Obvious.

(2)$\Rightarrow$(4) by Lemma 4.12, there is an invertible operator $y \in B(H)$ such that xy is unitary in $B(H)$, $< ye_k|e_n >= 0$ for $k < n$ and $< ye_n|e_n >> 0$ for $n \in N$. Obviously $< y^{-1}e_k|e_n >= 0$ for $k < n$ and $< y^{-1}e_n|e_n >> 0$ for $n \in N$. Since $(xy)^*xy = I, y^{-1} = (xy)^*x$ and y^{-1} is left injective with $\|y^{-1}\| = 1$. Hence by Lemma 4.13, $y^{-1} = I$ and x is unitary

(4)$\Rightarrow$(1) trivial.

Since x is left injective if and only if $x*$ is right injective, (3) $\Leftrightarrow$ (4) is trivial.

THEOREM 4.15. *Let H be a Hilbert space and $x \in B(H)$ be an isometry. Then x is left injective.*

Proof. Since x is an isometry, $xx^* = p$ is a projection and $xH = pH$ is closed. Hence there is a unitary $v : xH \longrightarrow H$. Define $U : xH \oplus xH^{\perp} \oplus$

$xH \oplus xH^{\perp} \longrightarrow xH \oplus xH^{\perp} \oplus xH \oplus xH^{\perp}$ with

$$U = \begin{pmatrix} pv & 0 & 0 & 0 \\ (I-p)v & 0 & 0 & 0 \\ 0 & 0 & xp & x(I-p) \\ 0 & I & 0 & 0 \end{pmatrix}.$$

Then U is unitary in $B(H \oplus H)$ and U is injective. Let $N \subset B(H)$ be injective. Then $\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}$ is injective in $B(H \oplus H)$. Since $U \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & xN \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & xN \end{pmatrix}$ is injective and xM is injective. Hence x is left injective.

REMARK 4.16. *Let x be an isometry but not unitary. Since xx^* is a projection with rank $p = \infty$ and $p \neq I$, p is not left injective. Hence x^* is not left injective, that is x is not right injective.*

REMARK 4.17. *Let A be a C^*-algebra. Then A has a unital imbedding in $B(H)$. Hence an isometry $x \in A$ is left injective.*

PROPOSITION 4.18. *Let H be an infinite dimensional Hilbert space and $x \in B(H)$ with finite rank. Then the following are equivalent:*

(1) *x is injective.*
(2) *x is left injective.*
(3) *x is right injective.*
(4) *rank $x = 0$ or 1.*

Proof. (1)($\Rightarrow$)(2) Obvious.

(2) $\Rightarrow$ (4) Suppose rank $x = k \geq 2$. Obviously rank $x^* = k$. Let $\{\alpha_1, ..., \alpha_k\} \subset \ker x^{\perp}$ and $\{\beta_1, ..., \beta_k\} \subset \operatorname{Ran} x$ be orthonormal bases respectly, K=Span $\{\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_k\}$ and $p \in B(H)$ be the projection with Ran $p = K$. Then $pxp = x$. Let q be a projection with $p \leq q$ and rank $q = k+1$. Then $qxq = pxp$ and $qxq : qH \to qH$ is not invertible and rank $qxq = k \geq 2$. Hence qxq is not left injective. Therefore x is not left injective.

(4) $\Rightarrow$ (1) Since rank $x = 0$ or 1, Ran x and Ran x^* are extremely injective. Hence x is injective. Since x is left injective if and only if x^* is right injective, (3) $\Leftrightarrow$ (4) is trivial.

References

1. M. D. Choi and E. G. Effros, *Injectivity and Operator Spaces*, J. Functional Analysis 24(1977), 156-209.
2. M. Hamana, *Injective Envelope of C*-algebra*, J. Math. Soc. Japan 31(1979), 181-197.
3. R. V. Kadison and J. R. Ringrose, Fundamentals of the Theory of Operator Algebras 1, Academic Press, New York, 1983.
4. V. I. Paulsen, Completely Bounded Maps and Dilations, John Wiley & Sons, Inc. , New York 1986.
5. Z. J. Ruan, *On Matricially Normed Spaces Associated with Operator Algebras*, Ph. D. thesis, University of California, Los Angeles (1987).
6. D. Y. Shin, S. G. Lee and S. J. Cho, *On Minimal E-projection of operator Spaces*, Comm. Korean Math. Soc. 4 (1989), No. 2, 349-356.
7. M. Takesaki, Theory of Operator Algebras 1, Springer-Verlag, New York, (1979).

+Department of Mathematics
Seoul National University
Seoul 151-742, Korea

++Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea