Comm. Korean Math. Soc. 5 (1990), No. 1, pp. 49~53

A REMARK ON THE HARDY–LITTLEWOOD–SOBOLEV–THEOREM

E. G. KWON

1. In the n–dimensional Euclidean space E^n, the maximal function $Mf(x)$ of an integrable function $f(x)$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|dy,$$

where $m(B(x,r))$ denotes the n–dimensional volume of the ball $B(x,r) = \{y \in E^n; |x-y| < r\}$ and $dy = dy_1 dy_2 \cdots dy_n$. Also the Riesz potentials are defined for $f(x)$ and $\alpha > 0$ by

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{E^n} |y|^{-n+\alpha} f(x-y)dy, \quad x \in E^n$$

with a constant $\gamma(\alpha) = \sqrt{\pi^n} 2^\alpha \frac{\Gamma(\alpha/2)}{\Gamma(n/2-\alpha/2)}$. See [1. p.117].

The Hardy–Littlewood–Sobolev theorem (of fractional integration) says that if $f(x) \in L^p(E^n)$, $1 < p < \infty$, and $0 < \alpha < n$, $1/q = 1/p - \alpha/n$ then

$$\|I_\alpha f\|_q \leq A_{p,q} \|f\|_p.$$

Here $\|f\|_p$ denotes the usual $L^p(E^n)$ norm of $f(x)$ and $A_{p,q}$ denotes a constant depending only on p and q (and n) [1. p.119]. Compared with the Bessel potentials, it is known that the Riesz potentials leads to less favourable behavior as $|x| \to \infty$ [1. p.131]. Also if $f(x) \in L^p(E^n)$ and $f(x)$ is continuous in a deleted neighborhood of 0 then by a successive use of the intermediate value theorem one verifies that

$$|x|^n |f(x)|^p \sim \int_{|x|}^{2|x|} \cdots \int_{|x|}^{2|x|} |f(y)|^p dy_1 \cdots dy_n,$$

Received July 20, 1989.

which tends to 0 as $|x| \to 0$. Our question on this point is that how much the L^q behavior of $I_\alpha f$ is affected by the decreasing rapidity of $f(x)$ as $|x| \to \infty$ or as $|x| \to 0$.

THEOREM. *Let $1 < p < \infty$, $0 < s \le \infty$, and $0 < \alpha < \beta < n$. Suppose that $f(x) \in L^p(E^n)$ and*

$$F_\beta(x) = \operatorname*{ess\,sup}_y |x-y|^\beta |f(x-y)| \in L^s(E^n),$$

then

$$\|I_\alpha f\|_q \le C_{\alpha,\beta,p} \|f\|_p^{1-\delta} \|F_\beta\|_s^\delta, \tag{1}$$

where

$$\delta = \alpha/\beta \ \text{ and } \ 1/q = (1-\delta)/p + \delta/s. \tag{2}$$

COROLLARY. *Let $0 < \alpha < \beta < n$, $1 < p < \infty$, $q(1 - \frac{\alpha}{\beta}) = p$, $f(x) \in L^p(E^n)$ and $|x|^\beta |f(x)|$ be essentially bounded. Then $I_\alpha f \in L^q(E^n)$.*

2. For the proof of Theorem we let

$$E = \{x\,;\, Mf(x) < \infty \ \text{ and } \ F_\beta(x) < \infty\}$$

and

$$t(x,f) = [F_\beta(x)/Mf(x)]^{1/\beta}.$$

Then we divide $\|I_\alpha f\|_q^q$ into two parts ;

$$\begin{aligned} \|I_\alpha f\|_q^q &= \int_{E_n} |I_\alpha f(x)|^q dx \\ &= [\int_{2|x| \le t(x,f)} + \int_{2|x| > t(x,f)}] \, |I_\alpha f(x)|^q dx \\ &= (I) + (II). \end{aligned} \tag{3}$$

First, to estimate (I) fix $x \in E$ such that $2|x| \leq t = t(x,f)$. Then since $|x-y| \geq \frac{|y|}{2}$ if $|y| > t$ in this case, we have

(4)
$$\begin{aligned}\int_{|y|>t} |y|^{-n+\alpha}|f(x-y)|dy &\leq F_\beta(x)\int_{|y|>t} |y|^{-n+\alpha}|x-y|^{-\beta}dy \\ &\leq 2^\beta F_\beta(x)\int_{|y|>t} |y|^{-n+\alpha-\beta}dy \\ &= 2^\beta(\beta-\alpha)^{-1}wt^{\alpha-\beta}F_\beta(x).\end{aligned}$$

Here w is the volume of the unit sphere $S^{n-1} = \{\zeta \in E^n : |\zeta| = 1\}$. On the other hand, if we temporarily set

$$\Omega(r) = \Omega(r,x) = r^{n-1}\int_{S^{n-1}} |f(x-r\zeta)|d\sigma(\zeta),$$

where $d\sigma$ is the element of volume on S^{n-1}, then by use of the integration by parts we obtain

(5)
$$\begin{aligned}&\int_{|y|\leq t} |y|^{-n+\alpha}|f(x-y)|dy \\ &= \int_0^t r^{-n+\alpha}\Omega(r)dr \\ &= t^{-n+\alpha}\int_0^t \Omega(r)dr + (n-\alpha)\int_0^t r^{-n+\alpha-1}\left[\int_{|y|<r} |f(x-y)|dy\right] dr \\ &\leq \alpha^{-1}nVt^\alpha Mf(x),\end{aligned}$$

where V is the volume of the unit ball $\{x\,;\,|x|<1\}$.

Combining (4) and (5), we can majorize (I) ;

$$(I) \leq A^q_{\alpha,\beta}\int_{2|x|\leq t(x,f)} Mf(x)^{q(1-\delta)}F_\beta(x)^{q\delta}dx, \tag{6}$$

where $A_{\alpha,\beta} = \frac{1}{\gamma(\alpha)}\left[\frac{nV}{\alpha} + \frac{2^\beta w}{(\beta-\alpha)}\right]$.

3. Next, it is not difficult to see from [1. p.118] that

$$\int_{E^n} |y|^{-n+\alpha}|x-y|^{-\beta}dy = \frac{\gamma(\alpha)\gamma(n-\beta)}{\gamma(n+\alpha-\beta)}|x|^{\alpha-\beta}.$$

Thus,

$$\begin{aligned}(7) \qquad (II) &= \gamma(\alpha)^{-q}\int_{2|x|>t}\left[\int_{E^n}|y|^{-n+\alpha}|f(x-y)|dy\right]^q dx \\ &\le \gamma(\alpha)^{-q}\int_{2|x|>t}F_\beta(x)^q\left[\int_{E^n}|y|^{-n+\alpha}|x-y|^{-\beta}dy\right]^q dx \\ &= \gamma(n-\beta)^q\gamma(n+\alpha-\beta)^{-q}\int_{2|x|>t}F_\beta(x)^q|x|^{-q(\beta-\alpha)}dx \\ &\le B^q_{\alpha,\beta}\int_{E^n}Mf(x)^{q(1-\delta)}F_\beta(x)^{q\delta}dx,\end{aligned}$$

where $B_{\alpha,\beta} = 2^{\beta-\alpha}\gamma(n-\beta)\gamma(n+\alpha-\beta)^{-1}$.

Therefore combining (6), (7), and (1),

$$\int_{E^n}|I_\alpha f(x)|^q dx \le C^q_{\alpha,\beta}\int_{E^n}Mf(x)^{q(1-\delta)}F_\beta(x)^{q\delta}dx,$$

where $C_{\alpha,\beta} = A_{\alpha,\beta} + B_{\alpha,\beta}$. Applying Hölder's inequality we finally obtain

$$(8) \qquad \|I_\alpha f\|_q \le C_{\alpha,\beta}\|Mf\|_p^{(1-\delta)}\|F_\beta\|_s^\delta.$$

Now the required result follows from the Maximal theorem [1. p.5].

4. Let us see that our exponents condition (2) on q and δ are appropriate. For the purpose assume (1) and change $f(x)$ with its dilation defined by $\tau_\nu f(x) = f(\nu x)$, $\nu > 0$. Noting that

$$\|I_\alpha(\tau_\nu f)\|_q = \nu^{-\frac{n}{q}-\alpha}\|I_\alpha f\|_q,$$
$$\|\tau_\nu f\|_p = \nu^{-\frac{n}{p}}\|f\|_p,$$

[1. p.118] and

$$\sup_y |x-y|^\beta |\tau_\nu f(x-y)| = \nu^{-\beta} F_\beta(\nu x) = \nu^{-\beta}\tau_\nu F_\beta(x),$$

we have by (1),

$$\nu^{-\frac{n}{q}-\alpha}\|I_\alpha f\|_q \le C_{\alpha,\beta,p}\nu^{-\frac{n(1-\delta)}{p}-\frac{n\delta}{s}-\beta s}\|f\|_p^{1-\delta}\|F_\beta\|_s^\delta$$

for all $\nu > 0$. Thus we should have

$$\frac{n}{q}+\alpha = \left[\frac{1-\delta}{p}+\frac{\delta}{s}\right] n + \beta\delta. \tag{8}$$

If (8) holds independently of n, then (8) is equivalent to (2).

References

1. E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, NJ, 1970.

Department of Mathematics Education
Andong National University
Andong 760–749, Korea