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TOPOLOGICAL MV–SEMIGROUPS

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In [2], Chae investigated basic theory of binary set–valued topological algebra and in particular, the properties of set–valued multiplications on an interval.

In this paper, topological multivalued–semigroups are defined and obtained results similar to those in [2].

1. Multifunctions

A multifunction $f : X \to Y$ is a correspondence from X to Y with $f(x)$ a nonempty subset of Y for each $x \in X$. We will denote the graph of f, i.e., $\{(x,y) : x \in X \text{ and } y \in f(x)\}$, by $G(f)$.

If $A \subset X$ and $B \subset Y$, we use the notation $f(A) = \cup\{f(x) : x \in A\}$, $f^{-1}(B) = \{x \in X : f(x) \cap B \neq \emptyset\}$ and $f^{[-1]}(B) = \{x \in X : f(x) \subset B\}$. We will denote the closure of a subset K of a topological space by $\overline{K}$.

We will say that a multifunction $f : X \to Y$ has closed (connected) [compact] point images if $f(x)$ is closed (connected)[compact] in Y for each $x \in X$. f is said to be a closed multifunction if $f(A)$ is closed in Y for all closed sets $A \subset X$.

Note. Let $f : X \to Y$ be a multifunction. Then the induced f^{-1} : $\mathcal{P}(Y) \to \mathcal{P}(X)$ preserves the elementary set operations. Precisely,

(1) $f^{-1}(\cup_\alpha B_\alpha) = \cup_\alpha f^{-1}(B_\alpha)$

(2) $f^{-1}(\cap_\alpha B_\alpha) \subset \cap_\alpha f^{-1}(B_\alpha)$

For the induced map $f : \mathcal{P}(X) \to \mathcal{P}(Y)$;

(1) $f(\cup_\alpha A_\alpha) = \cup_\alpha f(A_\alpha)$

(2) $f(\cap_a A_\alpha) \subset \cap_\alpha f(A_\alpha)$

For the combined action of f and f^{-1}, it is simple to veryfy

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THEOREM 1.1. *If $f : X \to Y$ is a multifunction, then:*
(1) For each $A \subset X$, $f^{-1}[f(A)] \supset A$ and $f^{[-1]}[f(A)] \supset A$
(2) $X - f^{-1}(K) = f^{[-1]}(Y - K)$ for each $K \subset Y$
(3) $f(f^{[-1]}(B)) \subset B$ for each $B \subset Y$.

Given $f : X \to Y$ and $g : Y \to Z$ are multifunctions, their composition $g \circ f : X \to Z$ is defined as the map $x \mapsto g(f(x))$. Then $g \circ f$ is also a multifunction.

We can clearly compose the induced maps f^{-1}, g^{-1} and we have

PROPOSITION 1.2. *Let $f : X \to Y$ and $g : Y \to Z$ be multifunctions. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Given an $f : X \to Y$ multifunction and a subset $A \subset X$, the f considered only on A is called the restriction of f to A, is written $f \mid A$, and can alternatively be defined as $f \mid A = f \cap (A \times Y)$.

2. Upper semi–continuous multifunctions

DEFINITION 2.1. *If X and Y are topological spaces and $f : X \to Y$ is a multifunction we will say that f has a closed graph if $G(f)$ is a closed subset of the product space $X \times Y$. If X and Y are topological spaces a multifunction $f : X \to Y$ is said to be upper semi–continuous at $x \in X$ if for each W open about $f(x)$ in Y there is a V open about x in X with $f(V) \subset W$: f is said to be upper semi–continuous if f is upper semi–continuous at each $x \in X$.*

EXAMPLE 2.2. *Let $I_u = [0,1]$ be the real unit interval with the usual topology. Define*

$$f(x) = \begin{cases} (0,x] & \text{if } 0 < x \leq 1 \\ \{0\} & \text{if } x = 0 \end{cases}$$

Then f is upper semi–continuous.

The elementary properties are

THEOREM 2.3. *(1) [9] (composition) If $f: X \to Y$ and $g: Y \to Z$ are upper semi-continuous, so also is $g \cdot f: X \to Z$.*

(2) (restriction of domain) If $f: X \to Y$ is upper semi-continuous and $A \subset X$ is taken with the subspace topology, then $f \mid A : A \to Y$ is upper semi-continuous.

(3) (Restriction of range) If $f: X \to Y$ is upper semi-continuous and $f(X)$ is taken with the subspace topology, then $f: X \to f(X)$ is upper semi-continuous.

The following lemma is a criterion for the upper semi-continuity of multifunctions.

LEMMA 2.4. *Let X, Y be topological spaces, and $f: X \to Y$ a multifunction. The following statements are equivalent.*

(1) f is upper semi-continuous.

(2) $f^{-1}(K)$ is closed in X whenever K is closed in Y.

(3) $f^{[-1]}(G)$ is open in X whenever G is open in Y.

(4) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every B Y.

(5) f^{-1} is a closed multifunction on $f(X)$ where $f^{-1}(y) = \{x \in X : y \in f(x)\}$.

Proof. (1) $\Leftrightarrow$ (2) [9].

(2) $\Rightarrow$ (4) : Let $B \subset Y$. Then $f^{-1}(B) \subset f^{-1}(\overline{B})$. By (2), $f^{-1}(\overline{B})$ is closed in X. Hence $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

(4) $\Rightarrow$ (2) : Let K be closed in Y. Then, by (4), $\overline{f^{-1}(K)} \subset f^{-1}(\overline{K}) = f^{-1}(K)$. Hence $f^{-1}(K)$ is closed in X.

(2) $\Leftrightarrow$ (3) : It is clear from theorem 1.1. (2).

(2) $\Rightarrow$ (5) : If F is closed in $f(X)$, then $F = f(X) \cap K$ for some closed subset K of Y. Since $f^{-1}(F) = f^{-1}(K)$ and $f^{-1}(K)$ is closed in X by (2), $f^{-1}(F)$ is closed in X.

(5) $\Rightarrow$ (2) : If K is closed in Y, then $K \cap f(X)$ is closed in $f(X)$. Since $f^{-1}(K) = f^{-1}(K \cap f(X))$ and $f^{-1}(K \cap f(X))$ is closed in X by (5), $f^{-1}(K)$ is closed in X.

REMARKS. *Suppose that X and Y are topological spaces and f is a function from X into Y. Then f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$*

for all $A \subset X$ [8]. The following examples show that this is not true if f is an upper semi–continuous multifunction.

EXAMPLE 2.5.

(1) Let $X = Y = [0,1]$ be the real unit interval with the usual topology and define f by : $f(x) = (1/2)x$ if $0 \leq x < 1/2$, $f(1/2) = [1/4,\ 3/4]$ and $f(x) = (1/2)(x+1)$ if $1/2 < x \leq 1$. If $A = [1/4,\ 1/2)$, then $f(\overline{A}) = [1/8,\ 3/4]$ and $\overline{f(A)} = [1/8,\ 1/4]$, i.e., $f(\overline{A}) \underset{\neq}{\supset} \overline{f(A)}$.

(2) Let $X = Y = [0,1]$ and define f by: $f(x) = [0,x]$ for $0 \leq x < 1$ and $f(1) = \{0\}$. Then $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$. But f is not upper semi–continuous because $f^{-1}([1/2,\ 1]) = [1/2,\ 1)$.

Compactness or connectedness is not in general preserved by upper semi–continuous multifunctions [9].

THEOREM 2.6 [9]. *Let $f : X \to Y$ be upper semi–continuous. If $f(x)$ is connected for each $x \in X$ and if $C \subset X$ is connected, then $f(C)$ is connected.*

THEOREM 2.7. [9]. *Let $f : X \to Y$ be an upper semi–continuous multifunction. If f has compact point images, then $f(K)$ is compact for any compact subset K of X.*

PROPOSITION 2.8. *Let $f : X \to Y$ be a function. Define $\overline{f} : X \to Y$ via $\overline{f}(x) = \{f(x)\}$. Then f is upper semi–continuous if and only if f is continuous.*

Proof. Let $A \subset Y$. Then $\{x \in X : \overline{f}(x) \cap A \neq \emptyset\} = f^{-1}(A)$. Hence, by lemma 2.4., $\overline{f}$ is upper semi–continuous if and only if f is continuous.

THEOREM 2.9. [10]. *If X is a topological space and the multifunction $f : X \to X$ has a closed graph, then $\{x \in X : x \in f(x)\}$ is closed in X.*

THEOREM 2.10. [4]. *If $f : X \to Y$ is an upper semi–continuous multifunction with closed point images and Y a regular space, then f has a closed graph.*

COROLLARY 2.11. *If X is a regular space and the upper semi–continuous multifunction $f : X \to X$ has closed point images, then $\{x \in X : x \in f(x)\}$ is closed in X.*

THEOREM 2.12. *Let $\{Y_\alpha : \alpha \in A\}$ be any family of topological spaces, and $f : X \to \Pi_a Y_\alpha$ a multifunction. Then f is upper semi–continuous if and only if $\overline{p}_\beta \circ f$ is upper semi–continuous for each $\beta \in A$ where $\overline{p}_\beta(y) = \{p_\beta(y)\}$ for each $y \in \Pi_\alpha Y_\alpha$.*

Proof. Let f be upper semi–continuous ; since p is upper semi–continuous, so also is $\overline{p}_\beta \circ f$.

Conversely, assume each $\overline{p}_\beta \circ f$ upper semi–continuous. If W is an open set containing $f(x)$, then $(\overline{p}_\beta \circ f)(x) \subset \overline{p}_\beta(W)$ and $\overline{p}_\beta(W)$ is open in Y_β. Since $p_\beta \circ f$ is upper semi–continuous, there exists U_β such that $x \in U_\beta$, $(\overline{p}_\beta \circ f)(U_\beta) \subset \overline{p}_\beta(W)$, U_β is open in X. But $p_\alpha(W) = Y$ for all but at most finitely many α, say $\alpha_1, \ldots, \alpha_n$. Put $U = \bigcap_{i=1}^{n} U_{\alpha_i}$. Then $f(U) \subset W$ and hence f is upper semi–continuous at $x \in X$.

3. Topological MV–semigroup

DEFINITION 3.1. *A topological MV–semigroup is a nonempty Hausdorff space S togethere with an upper semi–continuous multiplication $S \times S \to S$ (whose valued at (x, y) will be denoted by xy) satisfying $(xy)z = x(yz)$ for all $x, y, z \in S$. AB is defined to be the union $\cup\{ab : a \in A,\ b \in B\}$ for $A, B \subset S$.*

The condition that the multiplication on S is upper semi–continuous is equivalent to the condition that for each $x, y \in S$ and each open set W in S with $xy \subset W$, there exist open sets U and V in S such that $x \in U$, $y \in V$ and $UV \subset W$.

EXAMPLE 3.2.

(1) Any space X is a topological MV–semigroup under the multiplication $xy = \{x, y\}$ for each $x, y \in X$

(2) Let $X = [0, 1]$ be the real unit closed interval with the usual topol-

ogy. Then X is a topological MV–semigroup under the multiplication

$$xy = \begin{cases} (0, \min\{x,y\}] & \text{if } x \neq 0 \text{ and } y \neq 0 \\ \{0\} & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

(3) Let X be as in (2). Then X is a topological MV–semigroup under the multiplication

$$xy = \begin{cases} (0, \text{ the usual product } x \text{ and } y] & \text{if } x \neq 0 \text{ and } y \neq 0 \\ \{0\} & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

REMARKS. *It will be observed that no differentiation is made between x and $\{x\}$ if it is not convenient to do so and will not readily lead to confusion.*

PROPOSITION 3.3. *Let A and B be subsets of a topological MV–semigroup S.*

(a) If the multiplication of S has compact point images, then AB is compact for compact subsets $A, B \subset S$.

(b) If the multiplication of S has connected point images, then AB is connected for connected subsets $A, B \subset S$.

THEOREM 3.4. *Let A and B be compact subsets of a topological MV–semigroup S. If AB is contained in an open subset W of S, then there exist open subsets U and V of S such that $A \subset U$, $B \subset V$ and $UV \subset W$.*

Proof. Since $AB \subset W$, $ab \subset W$ for each $a \in A$ and each $b \in B$, there exist open sets M and N in S such that $a \in M$, $b \in N$, and $MN \subset W$. Since B is compact, for a fixed $a \in A$, there are open sets $M_1, \cdots, M_n$ in S containing a and corresponding open sets $N_1, \cdots, N_n$ in S such that $B \subset Q = N_1 \cup \cdots \cup N_n$. Let $P = M_1 \cap \cdots \cap M_n$. Then P is open in S, Q is open in S, $a \in P$, $B \subset Q$, and $PQ \subset W$. Since A is compact, there exist open sets $P_1, \cdots, P_m$ in S and corresponding $Q_1, \cdots, Q_m$ open in S such that $B \subset V = Q_1 \cap \cdots \cap Q_m$ and $A \subset U = P_1 \cup \cdots \cup P_m$. It follows that U and V are the required open sets.

COROLLARY 3.5. *Let A be a compact subset of a topological MV–semigroup and let $x \in S$. If $Ax[xA]$ is contained in an open subset V*

of S, there exsits an open subset U of S such that $x \in U$ and $AU \subset V[UA \subset V]$.

Notations. For subsets A and B of a topological MV–semigroup S, it is convenient to wirte

$$A^{[-1]}B = \{x \in S : Ax \subset B\}, \quad A^{(-1)}B = \{x \in S : Ax \cap B \neq \emptyset\}$$
$$BA^{[-1]} = \{x \in S : xA \subset B\}, \quad BA^{(-1)} = \{x \in S : xA \cap B \neq \emptyset\}.$$

In the case where S is a multi–mob, various forms of the proof of the following theorem have been given in [2].

THEOREM 3.6. *Let A and B be subsets of a topological MV– semigroup S. Then*
(1) If A is compact and if B is open, then $A^{[-1]}B$ is open.
(2) If A is compact and if B is closed, then $A^{(-1)}B$ is closed.
(3) If A is compact, then $\{x \in S : B \subset Ax\}$ is closed.

Proof. (1) If $x \in A^{[-1]}B$, then $Ax \subset B$. Since A is compact and since B is open, by corollary 3.5., there exists an open subset V of S such that $x \in V$ and $AV \subset B$, i.e., $V \subset A^{[-1]}B$. Therefore $A^{[-1]}B$ is open in S. (2) and (3) may be proved by observing that $A^{[-1]}(S-B) = S - A^{(-1)}B$ and $\{x \in S : B \subset Ax\} = \cap\{A^{(-1)}b : b \in B\}$ respectively.

DEFINITION 3.7. *Let S be a topological MV–semigroup.*
An element f of S is called a multi–idempotent if and only if $f \in f^2$.
An element u of S is called a left unit if and only if $x \in ux$ for each x in S.
An element S of S is called a left scalar if and only if sx is a singleton for each x in S.
An element u of S is called a left scalar unit if and only if u is a left scalar and a left unit, i.e., $ux = x$ for each element x in S.
In each definition, above, right and two–sided elements are defined analogously.

THEOREM 3.8. *If a topological MV–semigroup S is regular space and xy is closed in S for every $x, y \in S$, then the set E of all multi–idempotents of S is closed.*

Proof. Suppose there is an element x in $\overline{E}-E$, i.e., $x \in \overline{E}$ and $x \notin x^2$. Since S is regular, there exist open subsets U and V of S such that $x \in U$, $x^2 \in V$, and $U \cap V = \emptyset$. Since $x^2 \subset V$, by theorem 3.4., there exist open subsets V_1 and V_2 of S such that $x \in V_1 \cap V_2$ and $V_1V_2 \subset V$. Therefore $V_1V_2 \cap U = \emptyset$. Let $W = V_1 \cap V_2 \cap U$. Then $x \in W = W^0$ and $W^2 \cap W = \emptyset$. Since $x \in \overline{E}$, $W \cap E \neq \emptyset$, i.e., there is an element e in W such that $e \in e^2$. Then $e \in W \cap W^2 \neq \emptyset$, which is a contradiction.

4. Subsemigroups and ideals

Convention. Throughout this section, S with denote a topological MV–semigroup and E will denote the set of all multi–idempotents of S.

DEFINITION 4.1. *A nonempty subset A of S is called a subsemigroup of S if and only if $A^2 \subset A$.*

The intersection of a family of subsemigroups of S is a subsemigroup of S if it is nonempty.

DEFINITION 4.2. *A nonempty subset A of S is said to be a left (right, two–sided) ideal of S if and only if $SA \subset A(AS \subset A,\ AS \cup SA \subset A)$.*

Note that the union and the intersection (if it is nonempty) of any collection of left [right, two–sided] ideals of S is again a left [right, two–sided] ideal of S.

LEMMA 4.3. *Let $A \subset S$ and let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of subsets of S. Then $A(\cup\{A_\lambda : \lambda \in \Lambda\}) = \cup\{AA_\lambda : \lambda \in \Lambda\}$, $A(\cap\{A_\lambda : \lambda \in \Lambda\}) \subset \cap\{AA_\lambda : \lambda \in \Lambda\}$.*

PROPOSITION 4.4. *$(AB)C = A(BC)$ for each $A, B, C \subset S$.*

Proof. Let A, B and C be subsets of S. If $x \in (AB)C$, then there is an element y in AB and an element c in C such that $x \in yc$. Since $y \in AB$, there is an element a in A and an element b in B such that $y \in ab$. Then $x \in yc \subset (ab)c = a(bc) \subset A(BC)$, and $(AB)C \subset A(BC)$. Similarly, $(AB)C \subset A(BC)$ holds.

THEOREM 4.5. *If S is compact and xy is compact for each $x, y \in S$, then each left [right, two–sided] ideal of S contains a minimal left [right, two–sided] ideal which is closed.*

Proof. Let L be a left ideal of S and let $\mathcal{L}$ be the collection of all closed left ideals of S which are contained in L. If $a \in L$, then $Sa \subset SL \subset L$ and $S(Sa) = (SS)a \subset Sa$. It follows that Sa is a left ideal of S contained in L. Since S is compact, Sa is compact and hence closed in S. And it belongs to $\mathcal{L}$. Therefore, $\mathcal{L}$ is nonempty. $\mathcal{L}$ is partially ordered by set inclusion. Let $\mathcal{L}_0$ be a chain in $\mathcal{L}$. Since $\mathcal{L}_0$ is a collection of closed subsets of the compact space S with finite intersection property, $\cap\mathcal{L}_0 \neq \emptyset$. By the proceding note, $\cap\mathcal{L}_0 \in \mathcal{L}$. Therefore every chain in $\mathcal{L}$ is lower bounded. By Zorn's lemma, there is a minimal element L_0 in $\mathcal{L}$. Now let L_1 be a left ideal of S which is contained in L_0 and let $b \in L_1$. Then Sb is a closed left ideal of S and $Sb \subset L_1 \subset L_0$. Hence $L_1 = L_0$, i.e., L_0 is a minimal left ideal of S and is closed. Similar arguments hold for right and two sided ideals.

THEOREM 4.6. *The minimal ideal of S is unique.*

Proof. Let K_1 and K_2 be minimal ideals of S. Then $K_1 \cap K_2$ is an ideal of S since $\emptyset \neq K_1K_2 \subset K_1 \cap K_2$. Since K_1 and K_2 are minimal, $K_1 = K_1 \cap K_2 = K_2$.

Throughout, K will denote the minimal ideal of S.

THEOREM 4.7. *Let $\mathcal{M}_L(\mathcal{M}_R)$ denote the collection of all minimal left [right] ideals of S.*

(1) If $\mathcal{M}_L \neq \emptyset$ ($\mathcal{M}_R \neq \emptyset$), then S has the minimal ideal K.

(2) L_1, $L_2 \in \mathcal{M}_L$ and $L_1 \cap L_2 \neq \emptyset$ imply $L_1 = L_2$.
R_1, $R_2 \in \mathcal{M}_R$ and $R_1 \cap R_2 \neq \emptyset$ imply $R_1 = R_2$.

(3) $\cup\mathcal{M}_L \subset K$ and $\cup\mathcal{M}_R \subset K$.

Proof. (1) Let $L \in \mathcal{M}_L$ and let I be an ideal of S, then $S(IL) = (SI)L \subset IL$ and hence IL is a left ideal of S. Since $IL \subset SL \subset L \in \mathcal{M}_L$, $IL = L$. Therefore, $L = IL \subset IS \subset I$, i.e., all minimal left ideals are contained in each ideal of S. Hence $\emptyset \neq \cup\mathcal{M}_L \subset \cap\{I : IS \cup SI \subset I\} = K$. (2) and (3) are clear.

REMARKS. *Suppose S is a topological semigroup. Then the minimal ideal of S is the union of all minimal left [right] ideals of S. The following examples show that these are not true if S is a topological MV-semigroup.*

EXAMPLE 4.8. *(1) Let $S = [a,b]$. Then S is a topological MV-semigroup under the multiplication $xy = [a,b)$ if $a \leq y < b$ and $xy = [a,b]$ if $y = b$. In S, $[a,b)$ is the only minimal left ideal of S. On the other hand, the only minimal right ideal of S is S itself. Therefore $\cup\mathcal{M}_L = [a,b) \underset{\neq}{\subset} K = S$.*

By routine arguements, one may obtain

PROPOSITION 4.9. *Let $L(R,K)$ be a minimal left [right, two–sided] ideal of S. Then $L = Sa(R = aS,\ K = SaS]$ for each $a \in L$ $[a \in R,\ a \in K]$*

THEOREM 4.10. *If S is connected, xy is connected for each x, y in S and S has a left unit, then each ideal of S is connected.*

Proof. Let J be an ideal of S. Since S has a left unit, $x \in Sx$ and Sx is connected for each $x \in S$. Since $J = \cup\{x : x \in J\} \subset \cup\{Sx : x \in J\}$ and since $Sx \subset SJ \subset J$ for each $x \in J$, $J = \cup\{Sx : x \in J\}$. Let $y_0 \in J$. Then $y_0 S \subset J$ and hence $J = (\cup\{Sx : x \in J\}) \cup y_0 S$. Since $y_0 S$ is connected and since $y_0 x \subset Sx \cap y_0 S$ for each $x \in J$, J is connected.

DEFINITION 4.11. *For each subset A of S, $J_0(A)$ will denote the union of all ideals of S contained in A. If A contains no ideals of S, then $J_0(A) = \emptyset$. If $J_0(A)$ is nonempty, then it is clearly the unique largest ideal of S contained in A. $R_0(A)$ and $L_0(A)$ are defined analogously.*

THEOREM 4.12. *Let A be a subset of S. If A is open and if S is compact, then $J_0(A)$, $L_0(A)$, and $R_0(A)$ are open.*

Proof. If $x \in J_0(A)$, then $Sx \subset SJ_0(A) \subset J_0(A) \subset A$. Since S is compact and A is open, by using theorem 3.4., there are an open subset U of S such that $x \in U$ and $SU \subset A$. Again, since $xS \subset A$, there is an open subset V of S such that $x \in V$ and $VS \subset A$. Now, since $SxS \subset A$, there is an open subset W of S such that $x \in W$ and $SWS \subset A$. Let $M = U \cap V \cap W \cap A$. Then M is an open subset of S containing x. By lemma 4.3., $M \cup MS \cup SM \cup SMS$ is an ideal of S. Since $M \cup MS \cup SM \cup SMS \subset A \cup VS \cup SU \cup SWS \subset A$, $M \cup MS \cup SM \cup SMS \subset J_0(A)$. Therefore, $J_0(A)$ is open. Similar arguments hold for $L_0(A)$ and $R_0(A)$.

THEOREM 4.13. *Suppose S is compact. Then each proper ideal of S is contained in a maximal proper ideal of S and each maximal proper ideal is open.*

Proof. Let J be a proper ideal of S and let $a \in S - J$. Since S is compact and $S - \{a\}$ is open, by theorem 4.12., $J_0(S - \{a\})$ is a proper open ideal of S containing J. Therefore it is sufficient to consider only open proper ideals. Let $\mathcal{B}$ be the set of all proper open ideals of S containing J. Then $\mathcal{B}$ is nonempty. $\mathcal{B}$ is partially ordered by set inclusion. Since $J_0(S - \{a\}) \in \mathcal{B}$, by the Hausdorff Maximal Principle, there exists a maximal chain $\mathcal{C}$ in $\mathcal{B}$ containing $J_0(S - \{a\})$. Let $M = \cup\mathcal{C}$. Then M is a maximal open ideal of S containing J. If M is not proper, i.e., $M = S$, then, $\mathcal{C}$ is an open cover of S. Since S is compact, there exist $M_1, \ldots, M_n \in \mathcal{C}$ such that $M_1 \subset M_2 \subset \cdots \subset M_n$ and $S \subset \cup\{M_j : j = 1, \ldots, n\}$, and hence $S = M_n$ which contradicts the fact that M_n is a proper ideal of S. Therefore M is a maximal proper open ideal of S containing J.

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