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ON THE ORIENTATIONS AND THE SPINORIAL STRUCTURES ON VECTOR BUNDLES *

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In [4] the authors have proved that for every real vector bundle $V \to X$ with compact base X there is an orientation of $V \oplus V$ and $V \oplus V \oplus V \oplus V$ has a spinorial structure. In this paper, we shall prove a relation between the first Stiefel-Whitney class $\omega_1(V) \in H^1(X : Z/2)$ (Z = the set of integers) and an element of $H^1(X : SO(n))$ (Proposition 1) and some relations between the second Stiefel-Whitney class $\omega_2(V) \in H^2(X : Z/2)$ and the spinorial structure on V (Theorem 4).

In this paper by V we mean a real vector bundle $\pi : V \to X$ such that $\dim(V) = n$ and X is compact. An *orientation* on V is a function which assigns an orientation to each fiber of V, subject to the following local compatibility condition : There should exist a local system (U, φ) in X such that $h : U \times \mathbf{R}^n \approx \pi^{-1}(U) = V|_U$ with $h \mid x \times \mathbf{R}^n : x \times \mathbf{R}^n \to \pi^{-1}(x) = V_x$ $(x \in U)$ which is an orientation preserving homomorphism. That is, there should exist sections $s_1, \dots, s_n$ $U \to \pi^{-1}(U)$ so that the basis $s_1(x), \dots, s_n(x)$ $(x \in U)$ determines the required orientation of V_x ([5]).

A cocycle $\{g_{ji} | i, j \in \Lambda\}$ (Λ is an indexing set) of X is defined as follows : There exist an open cover $\{U_i | i \in \Lambda\}$ and continuous maps $g_{ji} : U_i \cap U_j \to G$ (a topological group) such that $g_{kj}(x) \cdot g_{ji}(x) = g_{ki}(x)$ for each $x \in U_i \cap U_j \cap U_k$ where $i, j, k \in \Lambda$. Two cocycles (U_i, g_{ji}) and (V_r, h_{sr}) are equivalent if there exist continuous maps $g_i^r : U_i \cap V_r \to G$ such that $g_j^s(x) \cdot g_{ji}(x) \cdot g_i^r(x)^{-1} = h_{sr}(x)$ for each $x \in U_i \cap U_j \cap V_r \cap V_s$. This relation is an equivalence relation denoted by " $\sim$ " ([2]). We put

$$H^1(X : G) = (\text{the set of all cocycles of } X)/\sim$$

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([2]).

PROPOSITION 1. *For a real vector bundle V $\omega_1(V) = 0$ if and only if there is an element α of $H^1(X : SO(n))$, whose image under the map $H^1(X : SO(n)) \to H^1(X : GL_n(\mathbf{R}))$ is the class of the bundle V.*

Proof. Since $\omega_1(V) = 0$ if and only if V is orientable ([3]), we shall prove that V is orientable if and only if there is an element $\alpha \in H^1(X : SO(n))$, whose image under the map $H^1(X : SO(n)) \to H^1(X : GL_n(\mathbf{R}))$ is the class of the bundle V.

We suppose that V is orientable. Then there is a local coordinate systems (U_i, φ_i) $(i \in \Lambda)$ such that

$$\varphi_{i,x} = \varphi_i|x : x \times \mathbf{R}^n = \mathbf{R}^n \to \pi^{-1}(x)$$

is an orientation preserving isomorphism, where $x \in U_i$. Therefore, for $x \in U_i \cap U_j$ $(i, j \in \Lambda)$ there exists an $g_{ji}(x) \in SO(n)$ such that $g_{ji}(x)\varphi_{i,x} = \varphi_{j,x}$. We define

$$g_{ji} : U_i \cap U_j \to SO(n)$$

then g_{ji} is continuous since $\varphi_{i,x}$ and $\varphi_{j,x}$ are continuous. It is clear that $g_{ii}(x)$ is the identity matrix in $SO(n)$ and $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$ for $x \in U_i \cap U_j \cap U_k$. Therefore $\{g_{ji} \,|\, i, j \in \Lambda\}$ is a cocycle of X. Hence we have the class α of $\{g_{ji}\}$ which is an element of $H^1(X : SO(n))$.

Conversely, we assume that there is an element $\alpha \in H^1(X : SO(n))$ such that the image of α under the map

$$H^1(X : SO(n)) \to H^1(X : GL_n(\mathbf{R}))$$

is the class of the bundle V. Then there is a cocycle $\{g_{ji}|i, j \in \Lambda\}$ which is in α. Thus each $x \in U_i \cap U_j$ $g_{ji}(x) \in SO(n)$, where $\{U_i|i \in \Lambda\}$ is an open cover of X. For each $i \in \Lambda$ we consider the trivial vector bundle $(E_i = U_i \times \mathbf{R}^n,\ \pi_i', U_i)$. In the topologically disjoint union $\bigcup_{i\in\Lambda} E_i$ we give the equivalence relation $e_i \sim e_j \iff e_j = g_{ji}(x)e_i$ where $x \in U_i \cap U_j$, $e_i \in E_{i,x}$ and $e_j \in E_{j,x}$. We put

$$E = \bigcup_{i\in\Lambda} E_i / \sim$$

which has the quotient topology. Then for each $x \in U_i \cap U_j$ since $g_{ji}(x) \in SO(n)$ the vector bundle (E, π', X) is orientable. Since the vector bundles V and E have the same cocycle $\{g_{ji} \mid i,j \in \Lambda\}$ $V \approx E$ and thus V is orientable.

Let $\mathbf{R}^n$ ($\mathbf{R}$ is the set of all real numbers) have the quadratic form $Q(x) = \sum x_i^2$, where $x \in \mathbf{R}^n$. For the spinorial group Spin(n) of this case we have the group exact sequence :

$$1 \to Z/2 \to \mathrm{Spin}(n) \xrightarrow{\rho} SO(n) \to 1$$

([1] or [2]).

LEMMA 2. *$\rho : Spin(n) \to SO(n)$ is a locally trivial fibration.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthogonal basis of $\mathbf{R}^n$ such that $Q(e_i) = 1$ for $1 \leq i \leq n$. By induction on $p (0 \leq p \leq n)$ we shall prove our assertion. We define a neighborhood $V_p (0 \leq p \leq n)$ of $1 \in SO(n)$ and a continuous map

$$s_p : V_p \to \mathrm{Spin}(n)$$

such that

(i) $V_0 = SO(n)$ and $\forall \alpha \in V_0$ $s_0(\alpha) = 1$
(ii) $s_{p+1}(\alpha) = (1 + t\omega)s_p(\alpha)$ where $t = \alpha(e_{p+1})$ and $\omega = \rho(s_p(\alpha))(e_{p+1})$
(iii) V_{p+1} is the subset of V_p defined by the condition $Q(t + \omega) \neq 0$.

Then $\alpha^{-1}\rho(s_p(\alpha))$ leaves $e_1, \dots, e_p$ fixed, where $\alpha \in V_p$. For example, for $\alpha \in V_1$ $\alpha^{-1}\rho(s_1(\alpha))e_1 = e_1$ because that $s_1(\alpha) = 1 + \alpha(e_1)e_1$ and $\rho(s_1(\alpha))e_1 = (1 + \alpha(e_1)e_1)e_1(1 + \alpha(e_1)e_1)^{-1} = \frac{1}{2}(1 + \alpha(e_1)e_1)e_1(1 - \alpha(e_1)e_1) = \alpha(e_1)$. We assume that $\alpha^{-1}\rho(s_{p-1}(\alpha))$ $(p \geq 2)$ leaves $e_1, \dots,$ e_{p-1} fixed, and we shall prove that $\alpha^{-1}\rho(s_p(\alpha))$ leaves $e_1, \dots, e_p$ fixed.

Since

$$\begin{aligned} s_p(\alpha) &= (1 + t\omega)s_{p-1}(\alpha) \\ &= (1 + \alpha(e_p)s_{p-1}(\alpha)e_p s_{p-1}(\alpha)^{-1})s_{p-1}(\alpha) \\ &= (1 - \alpha(e_p)e_p)s_{p-1}(\alpha) \end{aligned}$$

for e_i $(i = 1, 2, \ldots, p-1)$

$$\begin{aligned}
&\alpha^{-1}\rho(s_p(\alpha))e_i \\
&\quad = \alpha^{-1}((1-\alpha(e_p)e_p)s_{p-1}(\alpha)e_i s_{p-1}(\alpha)^{-1}(1-\alpha(e_p)e_p)^{-1}) \\
&\quad = \alpha^{-1}(\alpha(e_i))(1-\alpha(e_p)e_p)(1-\alpha(e_p)e_p)^{-1} \\
&\quad = e_i.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\alpha^{-1}\rho(s_p(\alpha))e_p \\
&\quad = \alpha^{-1}((1-\alpha(e_p)e_p)s_{p-1}(\alpha)e_p s_{p-1}(\alpha)^{-1}(1-\alpha(e_p)e_p)^{-1}) \\
&\quad = -\alpha^{-1}((1-\alpha(e_p)e_p)e_p(1-\alpha(e_p)e_p)^{-1}) \\
&\quad = -\frac{1}{2}\alpha^{-1}((e_p-\alpha(e_p))(1+\alpha(e_p)e_p)) \\
&\quad = e_p.
\end{aligned}$$

Hence $\rho(s_n(\alpha)) = \alpha$, and the map $\alpha \to s_n(\alpha)$ is a section of ρ defined on the neighborhood V_n. Hence

$$\begin{array}{ccc}
V_n \times \{1,-1\} & \longrightarrow & \rho^{-1}(V_n) \\
\Cup & & \Cup \\
(\alpha, \lambda) & \longrightarrow & \lambda s_n(\alpha)
\end{array}$$

is a homeomorphism. For each $\alpha \in SO(n)$ αV_n is a neighborhood of α and it is clear that $\rho^{-1}(\alpha V_n) \approx \alpha V_n \times \{1,-1\}$.

DEFINITION 3. *For each real vector bundle $\pi : V \to X$ (X is compact) a spinorial structure on V is an element $\beta \in H^1(X : Spin(n))$ such that the image of β under the map*

$$H^1(X : Spin(n)) \to H^1(X : SO(n)) \to H^1(X : GL_n(\mathbf{R}))$$

is the class of the bundle V.

Let $G_n(\mathbf{R}^N)$ $(n \leq N)$ be the Grassmann manifold of n-dimensional subspaces of $\mathbf{R}^N$. The canonical n-dimensional bundle γ_n^N on $G_n(\mathbf{R}^N)$ is the subbundle of the product bundle $(G_n(\mathbf{R}^N)\times\mathbf{R}^N,\ p,\ G_n(\mathbf{R}^N))$ with the total space consisting of the subspace of pairs $(W,x)\in G_n(\mathbf{R}^N)\times\mathbf{R}^N$ with $x\in W$. The inclusion map $G_n(\mathbf{R}^N)\to G_n(\mathbf{R}^{N+1})$ defines the injective limit space $\operatorname{inj\,lim} G_n(\mathbf{R}^N)=G_n(\mathbf{R}^\infty)=BO(n)$. Similarly, $\gamma^n=\operatorname{inj\,lim}\gamma_n^N$, that is, the canonical bundle $\gamma^n\to BO(n)$ is a real vector bundle with $\dim_{\mathbf{R}}(\gamma^n)=n$. For every real vector bundle $V\to X$ (X is compact) there exists a continuous function $f:X\to BO(n)$ such that $V\cong f^*(\gamma^n)$ ([1], [3]). Moreover for the second Stiefel-Whitney class $\omega_2(\gamma^n)\in {}_cH^2(BO(n);Z_2)$ $f^*(\omega_2(\gamma^n))=\omega_2(V)\in {}_cH^2(X:Z_2)$ is the second Stiefel-Whitney class of V, where ${}_cH^i(Y:G)$ is the i^{th} Čeck cohomology group of the topological space Y with coefficients in a topological group G.

THEOREM 4. *Let a real vector bundle $V\to X$ (X is compact) be an oriented bundle.*

(i) There is a spinorial structure on V if and only if $\omega_2(V)\in {}_cH^2(X:Z_2)$ is zero.

(ii) If $\omega_2(V)=0$ there are 4 different spinorial structures on V.

Proof. (i) Note that ${}_cH^1(BO(n);Z_2)(\cong Z_2)$ has the generator $\omega_1(\gamma^n)$ and that ${}_cH^2(BO(n):Z_2)(\cong Z_2)$ has the generator $\omega_2(\gamma^n)$ ([3], [5]).

As before, we have a bundle morphism $(f,\overline{f}):(V\to X)\to(\gamma^n\to BO(n))$ such that

$$\begin{array}{ccc} V & \xrightarrow{\overline{f}} & \gamma^n \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BO(n) \end{array}$$

is commutative, $f^*(\gamma^n)\cong V$ and $f^*(\omega_i(\gamma^n))=\omega_i(V)$ for $i=1,2$. By Lemma 2, there exists a finer open cover $\{U_i\,|\,i\in\Lambda\}$ of X with a cocycle $\{g_{ji}:U_i\cap U_j\to SO(n)|i,j\in\Lambda\}$ of the bundle V (Note that V is an oriented bundle) such that

$$\rho^{-1}(g_{ji}(U_i\cap U_j))\cong g_{ji}(U_i\cap U_j)\times\{0,1\}\subset \mathrm{Spin}(n)$$

for $i,j\in\Lambda$ (recall that $\rho:\mathrm{Spin}(n)\to SO(n)$ is locally trivial). Then we have continuous maps

$$\begin{array}{rccc} \tilde{g}_{ji}: & U_i \cap U_j & \longrightarrow & \mathrm{Spin}(n) \\ & \Cup & & \Cup \\ & x & \longrightarrow & \tilde{g}_{ji}(x) = (g_{ji}(x), t_{ji}) \end{array}$$

for $i, j \in \Lambda$ and $t_{ji} = 1$ or 0. Moreover we define $g_{ji}(x) = 1_{n\times n}$ for all $x \in U_i$, where $1_{n\times n}$ is the unit matrix in $SO(n)$. We define continuous maps

$$\begin{array}{rccc} h_{ijk}: & U_i \cap U_j \cap U_k & \longrightarrow & \mathrm{Spin}(n) \\ & \Cup & & \Cup \\ & x & \longrightarrow & \tilde{g}_{ij}(x) \cdot \tilde{g}_{jk}(x) \cdot \tilde{g}_{ki}(x) \\ & & & = (1_{n\times n}, t_{ij} + t_{jk} + t_{ki}) \end{array}$$

for $i, j, k \in \Lambda$. Then it follows that $\rho(\tilde{g}_{ji}) = g_{ji}$. Let δ be the boundary operator in Čeck chain complex. Then

$$\begin{aligned} \delta h_{ijkl}(x) &= h_{jkl}(x) + h_{ikl}(x)^{-1} + h_{ijl}(x) + h_{ijk}(x)^{-1} \\ &= (1_{n\times n}, 0) \end{aligned}$$

for $x \in U_i \cap U_j \cap U_k \cap U_l$. Therefore $\{h_{ijk}\}$ determines an element of ${}_cH^2(X : \mathrm{Spin}(n))$.

We define a group homomorphism

$$\begin{array}{rccc} \phi: & \mathrm{Spin}(n) & \longrightarrow & Z_2 \\ & \Cup & & \Cup \\ & (A, t) & \longrightarrow & |A|t = t, \end{array}$$

where $A \in SO(n)$, $|A|$ is the determinant of A and $t = 1$ or 0. We put the cohomology class of $\{\phi h_{ijk}\} = \omega_2'(V) \in {}_cH^2(X : Z_2)$.
Since ${}_cH^2(BO(n) : Z_2) \cong Z_2$ we can put

$$f^*(\omega_2(\gamma^2)) = \omega_2(V) = \omega_2'(V).$$

It is clear that $\omega(V) = 0 = \omega'(V)$ if and only if $\{\tilde{g}_{ji}\}$ is a cocycle of the bundle V (see proof of (ii) below). Therefore $\omega_2(V) = 0$ if and only if there is a spinorial structure on V.

(ii) We shall use the notations in the proof of (i).

By the definition of ϕ we see that

$$\omega_2'(V) = 0 \Longleftrightarrow \text{for every } \{i,j,k\} \subset \Lambda \quad t_{ij} + t_{jk} + t_{ki} \equiv 0 \text{ mod} 2.$$

Therefore we have 4 cases :

(a) $t_{ij} = t_{jk} = t_{ki} = 0$, (b) $t_{ij} = t_{jk} = 1$ and $t_{ki} = 0$,
(c) $t_{ij} = t_{ki} = 1$ and $t_{jk} = 0$, (d) $t_{jk} = t_{ki} = 1$ and $t_{ij} = 0$.

They satisfy that, for example,

$$\tilde{g}_{ij}(x) \cdot \tilde{g}_{jk}(x) = \tilde{g}_{ik}(x) \quad (x \in U_i \cap U_j \cap U_k)$$

and

$$\tilde{g}_{ij}(x) = \tilde{g}_{ji}(x)^{-1}.$$

That is, in the above cases $\{\tilde{g}_{ji}\}$ is a cocycle of the bundle V. Thus if $\omega_2(V) = 0$ there are 4 different spinorial structures on V.

References

1. D. Husemoller, Fibre Bundles, Springer-Verlag,1966.

2. M. Karoubi, K-theory, Springer-Verlag,1978.

3. K. Lee, Fundation of Topology, Vol.2, Harkmunsa,1983.

4. K. Lee and H. Lee, *A note on spinorial structures in vector bundles,* Honam Math. Journal Vol.11 (1989).

5. J.W. Milnor, Characteristic Classes, Princeton Univ. Press,1974.

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