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APPLICATIONS OF THE GENERALIZED EVALUATION SUBGROUPS ON CONVERSES OF THE LEFSCHETZ FIXED POINT THEOREM

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1. Introduction.

Let X be a compact connected polyhedron and let $f : X \to X$ be a self map of X. Let $MF(f)$ stand for the least number of fixed points of self maps homotopic to f, $N(f)$ the Nielsen number of f, and $L(f)$ the Lefschetz number of f. We always have $N(f) \leq MF(f)$.

The celebrated Lefschetz fixed point theorem says that $L(f) \neq 0$ implies that every map homotopic to f has a fixed point, i.e., $MF(f) > 0$. Its converse statement, "$L(f) = 0$ implies $MF(f) = 0$" is not always true even for homeomorphisms of closed manifolds, as shown by example in [Mc]. It is desirable to understand under what restrictions on the space or the self map the converse does hold true.

In [J_1, J_2], Jiang showed the following theorem as a converse of the Lefschetz fixed point theorem:

THEOREM. *Let X be a compact connected polyhedron without gloval separating points. Suppose X satisfies the condition $\pi_1(X, x_0) = G(X, x_0)$ $(= J(X))$. Then the Lefschetz number $L(f) = 0$ iff f is homotopic to a fixed point free map.*

THEOREM. *Let X be a compact connected polyhedron without global separating points. Suppose $\pi_1(X, x_0)$ is finite and the universal covering space $\tilde{X}$ has the same rational homology as X. Then for any $f : X \to X$, $L(f) = 0$ iff f is homotopic to a fixed point free map.*

It is also desirable to understand under what restritions on the space which does not satisfy $\pi_1(X, x_0) = G(X, x_0)$ the converse does hold true.

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The purpose of this paper is to give a partial solution of the above question using the generalized evaluation subgroups of the fundamental group.

2. Notation and terminology.

Let X be a topological space with x_0 as a base point. A homotopy $H : X \times I \to X$ is called a *cyclic homotopy* [Go] if

$$H(x,0) = H(x,1) = x.$$

In another notation, h_t is a cyclic homotopy if $h_0 = h_1 = 1_X$, where 1_X denotes the identity map of X. If h_t is a cyclic homotopy, the path given by $\alpha : I \to X$ such that $\alpha(t) = h_t(x_0)$ is called the *trace* of h_t.

The set of homotopy classes of those loops which are the trace of some cyclic homotopy form a subgroup $G(X, x_0)$ of the fundamental group which is called the evaluation subgroup [Go].

In [WK], the author and Kim defined the generalized evaluation subgroup $G(X, A, x_0)$ of the fundamental group as follows; Let (X, A) be a topological pair and $i : A \to X$ be the inclusion. Consider the class of continuous functions $H : A \times I \to X$ such that

$$H(x,0) = H(x,1) = i(x).$$

Then the map $h : I \to X$ defined by $h(s) = H(x_0, s)$ represents an element $[h]$ in $\pi_1(X, x_0)$. The set of all elements $[h] \in \pi_1(X, x_0)$ obtained in the above manner from some H is denoted by $G(X, A, x_0)$. Thus for every $[h] \in G(X, A, x_0)$, there is at least one map $H : A \times I \to X$ such that $[H(x_0, \ )] = [h]$. H is called an *affiliated map* to $[h]$ with respect to A.

Let A be locally compact and regular, and X^A be the space of mappings from A to X with compact open topology. The map $p : X^A \to X$ given by $p(g) = g(x_0)$ is continuous. Thus p induces a homomorphism

$$p_* : \pi_1(X^A, i) \to \pi_1(X, x_0).$$

In this case, the image of p_* is $G(X, A, x_0)$. Thus $G(X, A, x_0)$ is called the *generalized evaluation subgroup* of the fundamental group $\pi_1(X, x_0)$.

It is easy to show that $J(X) = G(X)$ is a subgroup of $G(X, A)$.

3. Main results:

In the following theorem, we substitute the generalized evaluation subgroup for the evaluation subgroup in the converses of the Lefschetz fixed point theorem.

THEOREM 1. *Let X be a compact connected polyhedron without global separating points. Suppose there exists a compact connected subpolyhedron A of X such that A satisfies $\pi_1(X, x_0) = G(X, A, x_0)$ and also satisfies either of the following:*

(1). if X has no local separating points, $G(X, A, x_0)$ is abelian,

or

(2). if X has a local separating point, A has a deformation retract homeomorphic to S^1.

Then for any map $f : X \to X$ such that $f(X) \subset A$, $L(f) = 0$ iff f is homotopic to a fixed point free map.

Proof. Case 1. If X has no local separating points, then $\pi_1(X, x_0)$ is an abelian group by the hypothesis. For any map $f : X \to X$ such that $f(X) \subset A$, we have

$$\pi_1(X, x_0) = G(X, A, x_0) \subset G(X, f(X), x_0).$$

Thus $G(X, f(X), x_0) = \pi_1(X, x_0)$. Let $[h]$ be any element of $G(X, f(X), x_0)$. Then there exists a homotopy $H : f(X) \times I \to X$ such that

$$H(\ ,0) = i = H(\ ,1) \quad \text{and} \quad H[x_0,\ ] = [h].$$

Since $x_0 \in f(X)$, there exists an element $z \in X$ such that $f(z) = x_0$. Define a homotopy $K : X \times I \to X$ by $K = H(f\ x\ 1)$.
Then K is a continuous function and

$$\begin{aligned} K(x,0) &= H(f(x),0) = i\ f(x) = f(x), \\ K(x,1) &= H(f(x),1) = i\ f(x) = f(x), \\ K(z,t) &= H(f(z),t) = H(x_0,t) = h(t). \end{aligned}$$

Thus we have $[h] \in J(f, z)$. This means that $f_*(\pi_1(X, z)) \subset J(f, z)$. By Theorem 2.4.2 [J$_2$], we obtain that $L(f) = 0$ implies $N(f) = 0$.

Since $\pi_1(X, x_0)$ is an abelian group, X is not a surface of negative Euler characteristic. Therefore, X is a compact connected polyhedron and not a surface of negative Euler characteristic. If we use Theorem 1.6.3 [J_2], we have $MF(f) = N(f)$. By these two results, we have that $L(f) = 0$ implies $MF(f) = 0$. The converse is clear.

Case 2. Let X be a compact connected polyhedron with a local separating point which is not a global separating point. Since X has a compact connected subpolyhedron A which has a deformation retract homeomorphic to S^1, we have the inclusion $i : S^1 \to A$ and the deformation retraction $r : A \to S^1$. Let $f : X \to X$ be a self map such that $f(X) \subset A$ and $f_A : A \to A$ be its restriction. Consider

$$\begin{array}{cccc} g = r \circ f_A \circ i : & S^1 & \longrightarrow & S^1 \\ & \downarrow & & \downarrow \\ & A & \longrightarrow & A \end{array}$$

then g and f_A are of the same homotopy type. By homotopy type invariance of the Nielsen number (Theorem 1.5.3 [J_2]), we have $N(g) = N(f_A)$.

For S^1, we know that g can be homotoped to a map k with exactly $N(g)$ fixed points. Thus, on A, the map f_A (homotopic to $i \circ g \circ r$) can be homotoped to $i \circ k \circ r$ with exactly $N(g)$ fixed points. We denote this homotopy by G. Consider that A is an ANR and $H' : (X \times 0) \cup (A \times I) \to A$ such that $H'_{X \times 0} = f$, $H'_{A \times I} = G$, there exists a homotopy $H : X \times I \to A$ such that $H = H'$ on $(X \times 0) \cup (A \times I)$. Let $f' = i \circ H(\ ,1) : X \to X$. Then $f'(X) \subset A$ and $f'_A = i \circ k \circ r$. Since (X, A) is a pair of compact connected polyhedron and $f : X \to X$ satisfies $f(X) \subset A$, we have $N(f_A) = N(f)$ (Corollary 1.5.5 [J_2]).

Now f' and $f'_A = i \circ k \circ r$ have the same fixed points. Thus f' has exactly $N(f'_A)$ fixed points. Since f is homotopic to f', we have $MF(f) \leq \#\ \mathrm{Fix}(f') = \#\ \mathrm{Fix}(k) = N(g) = N(f_A) = N(f)$.

Otherwise, $N(f) \leq MF(f)$ is clear. Thus we have $MF(f) = N(f)$. By case 1, we already know that $L(f) = 0$ implies $N(f) = 0$. Therefore we have that $L(f) = 0$ iff $MF(f) = 0$.

THEOREM 2. *Let X be a compact connected polyhedron without global separating points. Suppose $\pi_1(X, x_0)$ is finite and X has a subspace A such that $G(X, A, x_0) = \pi_1(X, x_0)$. Then for any map $f : X \to$*

X such that $f(X) \subset A$, we have $L(f) = 0$ iff f is homotopic to a fixed point free map.

Proof. Since $\pi_1(X, x_0)$ is finite, X can not be a surface of negative Euler characteristic and X can not have a local separating point which is not a global one (Lemma 2.6.4 [J$_2$]). So, according to Theorem 1.6.3 [J$_2$], we have $N(f) = MF(f)$ for any $f : X \to X$. Since $G(X, A, x_0) = \pi_1(X, x_0)$, we know that $L(f) = 0$ implies $N(f) = 0$. Thus we obtain the result.

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