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ON CERTAIN CLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=p}^{\infty}a_nz^n,\quad (a_n\geq 0\,;\,p\in N=\{1,2,\dots\}) \tag{1.1}$$

which are analytic and univalent in $D=\{z:0<|z|<1\}$ with a simple pole at the origin with residue 1 there.

A function $f(z)$ in $\sum_p$ is said to be a member of $\sum_p(A,B)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)}+1\right|<\left|A+B\frac{zf'(z)}{f(z)}\right| \tag{1.2}$$

for $-1\leq A<B\leq 1,\ 0<B\leq 1$ and $z\in D$.

In particular, the class $\sum_1(-1,1)$ was studied by Padmanabhan [5] and the class $\sum_1(A,B)$ when $A=\beta(2\alpha-I)$ and $B=\beta(0\leq\alpha<1$ and $0<\beta\leq 1)$ were studied by Mogra, reddy, and Juneja [4].

The aim of the present paper is to investigate coefficient estimates, distortion properties and radius of convexity for the class $\sum_p(A,B)$. Furthermore, it is shown that the class $\sum_p(A,B)$ is closed under convex linear combinations, convolutions and integral transforms.

2. Coefficient estimates

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THEOREM 1. *Let* $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$, $a_n \geq 0$, *be regular in* D. *Then* $f(z)$ *is in the class* $\sum_p(A,B)$ *if and only if*

$$\sum_{n=p}^{\infty}\{(n+1)+(A+Bn)\}a_n \leq B-A, \tag{2.1}$$

for $-1 \leq A < B \leq 1$ *and* $0 < B \leq 1$.

Proof. Suppose that $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$, $a_n \geq 0$, is in $\sum_p(A,B)$. Then

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{A+B\frac{zf'(z)}{f(z)}}\right| = \left|\frac{\sum_{n=p}^{\infty}(n+1)a_n z^n}{(B-A)\frac{1}{z} - \sum_{n=p}^{\infty}(A+Bn)a_n z^n}\right| < 1$$

for all $z \in D$. Since $\mathrm{Re}(z) \leq |z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=p}^{\infty}(n+1)a_n z^n}{(B-A)\frac{1}{z} - \sum_{n=p}^{\infty}(A+Bn)a_n z^n}\right\} < 1, \ (z \in D). \tag{2.2}$$

Now choose the values of z on real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through positive values, we obtain

$$\sum_{n=p}^{\infty}\{(n+1)+(A+Bn)\}a_n \leq B-A.$$

Conversely, suppose that (2.1) holds for all admissible values of A and B. Then we have

$$\begin{aligned} H(f,f') &= |zf'(z)+f(z)| - |Af(z)+Bzf'(z)| \\ &= |\sum_{n=p}^{\infty}(n+1)a_n z^n| - |(B-A)\frac{1}{z} - \sum_{n=p}^{\infty}(A+Bn)a_n z^n| \end{aligned}$$

or

$$
\begin{aligned}
&|z|H(f,f') \\
&\le \sum_{n=p}^{\infty}(n+1)a_n|z|^{n+1} - (B-A) + \sum_{n=p}^{\infty}(A+Bn)a_n|z|^{n+1} \\
&= \sum_{n=p}^{\infty}\{(n+1)+(A+Bn)\}a_n|z|^{n+1} - (B-A).
\end{aligned}
$$

Since the above inequality holds for all $r = |z|,\ 0 < r < 1$, letting $r \to 1$, we have

$$\sum_{n=p}^{\infty}\{(n+1)+(A+Bn)\}a_n \le B-A$$

by (2.1). Hence it follows that $f(z)$ is in the class $\sum_p(A,B)$.

COROLLARY. *If the function* $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ *is in the class* $\sum_p(A,B)$, *then we have*

$$a_n \le \frac{B-A}{(n+1)+(A+Bn)}, \quad (n \ge p). \tag{2.3}$$

The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{B-A}{(n+1)+(A+Bn)}z^n, \quad (n \ge p). \tag{2.4}$$

3. Distortion properties and radius of convexity

THEOREM 2. *If the function* $f(z)\frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ *is in the class* $\sum_p(A,B)$, *then we have*

$$\frac{1}{|z|} - \frac{B-A}{(p+1)+A+Bp}|z|^p \le |f(z)| \le \frac{1}{|z|} + \frac{B-A}{(p+1)+A+Bp}|z|^p.$$

The result is sharp.

Proof. Suppose that $f(z)$ is in $\sum_p(A,B)$. By Theorem 1, we have

$$\sum_{n=p}^{\infty} a_n \leq \frac{B-A}{(p+1)+A+Bp}.$$

Thus

$$\begin{aligned}|f(z)| &\leq \frac{1}{|z|} + |z|^p \sum_{n=p}^{\infty} a_n \\ &\leq \frac{1}{|z|} + \frac{B-A}{(p+1)+A+Bp}|z|^p.\end{aligned}$$

Also,

$$\begin{aligned}|f(z)| &\geq \frac{1}{|z|} - |z|^p \sum_{n=p}^{\infty} a_n \\ &\geq \frac{1}{|z|} - \frac{B-A}{(p+1)+A+Bp}|z|^p.\end{aligned}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{B-A}{(p+1)+A+Bp}z^p.$$

THEOREM 3. *If the function* $f(z) = \dfrac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ *is in the class* $\sum_p(A,B)$, *then* $f(z)$ *is meromorphically convex of order* $\delta (0 \leq \delta < 1)$ *in* $|z| < r = r(A,B,\delta)$, *where*

$$r(A,B,\delta) = \inf_{n \geq p} \left\{ \frac{(1-\delta)(n+1)+(A+Bn)}{(B-A)n(n+2-\delta)} \right\}^{\frac{1}{n+1}}$$

The result is sharp.

Proof. Let $f(z)$ is in $\sum_p^*(A,B)$. Then, by Theorem 1, we have

$$\sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n \leq 1. \tag{3.1}$$

It is sufficient to show that

$$|2+\frac{2f''(z)}{f'(z)}| \leq 1-\delta$$

for $|z| \leq r(A,B,\delta)$, where $r(A,B,\delta)$ is as specified in the statement of the theorem. Then

$$\begin{aligned} |2+\frac{zf''(z)}{f'(z)}| &= \left|\frac{\sum_{n=p}^{\infty} n(n+1)a_n z^{n-1}}{-\frac{1}{z^2}-\sum_{n=p}^{\infty} na_n z^{n-1}}\right| \\ &\leq \frac{\sum_{n=p}^{\infty} n(n+1)a_n|z|^{n+1}}{1-\sum_{n=p}^{\infty} na_n|z|^{n+1}}. \end{aligned}$$

This will be bounded by $1-\delta$ if

$$\sum_{n=p}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n|z|^{n+1} \leq 1. \tag{3.2}$$

By (3.1), it follows that (3.2) is true if

$$\frac{n(n+2-\delta)}{1-\delta}|z|^{n+1} \leq \frac{(n+1)+(A+Bn)}{B-A}, \quad (n \geq p)$$

or

$$|z| \leq \left\{\frac{(1-\delta)\{(n+1)+(A+Bn)\}}{(B-A)n(n+2-\delta)}\right\}^{\frac{1}{n+1}}, \quad (n \geq p). \tag{3.3}$$

Setting $|z| = r(A,B,\delta)$ in (3.3), the result follows. The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{B-A}{(n+1)+(A+Bn)} z^n, \ (n \geq p). \tag{3.4}$$

REMARK. *A function $f(z) \in \sum_p$ is said to be meromorphically convex of order $\delta(0 \leq \delta < 1)$ if*

$$Re\{-(1 + \frac{zf''(z)}{f'(z)})\} > \delta, \quad (z \in D).$$

4. Convex linear combinations and convolution properties

THEOREM 4. *Let $f_0(z) = \frac{1}{z}$ and*

$$f_n(z) = \frac{1}{z} + \frac{B - A}{(n+1) + (A + Bn)} z^n, \quad (n \geq p).$$

Then $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ is in the class $\sum_p(A, B)$ if and only if it can be expressed in the form

$$f(z) = \lambda_0 f_0(z) + \sum_{n=p}^{\infty} \lambda_n f_n(z),$$

where $\lambda_0 \geq 0,\ \lambda_n \geq 0\ (n \geq p)$ and $\lambda_0 + \sum_{n=p}^{\infty} \lambda_n = 1$.

Proof. Let $f(z) = \lambda_0 f_0(z) + \sum_{n=p}^{\infty} \lambda_n f_n(z)$ with $\lambda_0 \geq 0,\ \lambda_n \geq 0\ (n \geq p)$ and $\lambda_0 + \sum_{n=p}^{\infty} \lambda_n = 1$. Then

$$\begin{aligned} f(z) &= \lambda_0 f_0(z) + \sum_{n=p}^{\infty} \lambda_n f_n(z) \\ &= \frac{1}{z} + \sum_{n=p}^{\infty} \lambda_n \frac{B - A}{(n+1) + (A + Bn)} z^n. \end{aligned}$$

Since

$$\sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} \lambda_n \frac{B-A}{(n+1)+(A+Bn)} = \sum_{n=p}^{\infty} \lambda_n$$
$$= 1 - \lambda_0 \leq 1,$$

by Theorem 1, $f(z)$ is in the class $\sum_p(A,B)$.

Conversely, suppose that the function $f(z)$ is in the class $\sum_p(A,B)$. Since

$$a_n \leq \frac{B-A}{(n+1)+(A+Bn)}, \quad (n \geq p),$$

setting

$$\lambda_n = \frac{(n+1)+(A+Bn)}{B-A} a_n, \quad (n \geq p),$$

and

$$\lambda_0 = 1 - \sum_{n=p}^{\infty} \lambda_n,$$

it follows that $f(z) = \lambda_0 f_0(z) + \sum_{n=p}^{\infty} \lambda_n f_n(z)$. This completes the proof of the theorem.

For the functions $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=p}^{\infty} b_n z^n$ belonging to $\sum_p$, we denote by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$, or

$$(f * g)(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n b_n z^n.$$

THEOREM 5. *If the functions* $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ *and* $g(z) = \frac{1}{z} + \sum_{n=p}^{\infty} b_n z^n$ *are in the class* $\sum_p(A,B)$, *then* $(f * g)(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n b_n z^n$ *is in the class* $\sum_p(A,B)$.

Proof. Suppose that $f(z)$ and $g(z)$ are in $\sum_p(A,B)$. By Theorem 1, we have

$$\sum_{n=p}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}a_n\le 1$$

and

$$\sum_{n=p}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}b_n\le 1.$$

Since $f(z)$ and $g(z)$ are regular in D, so is $(f*g)(z)$. Furthermore,

$$\begin{aligned}&\sum_{n=p}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}a_nb_n\\ \le&\sum_{n=p}^{\infty}\left\{\frac{(n+1)+(A+Bn)}{B-A}\right\}^2a_nb_n\\ \le&\left[\sum_{n=p}^{\infty}\left\{\frac{(n+1)+(A+Bn)}{B-A}\right\}a_n\right]\left[\sum_{n=p}^{\infty}\left\{\frac{(n+1)+(A+Bn)}{B-A}\right\}b_n\right]\\ \le& 1.\end{aligned}$$

Hence by Theorem 1, $(f*g)(z)$ is in the class $\sum_p(A,B)$.

5. Integral transforms

In this sectionwe consider integral transforms of functions in $\sum_p(A,B)$ of the type considered by Bajpai [1] and Goel and Sohi [3].

THEOREM 6. *If the function* $f(z)=\dfrac{1}{z}+\sum_{n=p}^{\infty}a_nz^n$ *is in the class* $\sum_p(A,B)$, *then the integral transforms*

$$F_c(z)=c\int_0^1u^cf(uz)du,\quad(0<c<\infty)$$

are in the class $\sum_p(A,B)$.

Proof. Suppose that $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ is in $\sum_p(A, B)$. Then we have

$$F_c(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z} + \sum_{n=p}^{\infty} \frac{ca_n}{n+c+1} z^n.$$

Since

$$\sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} \frac{ca_n}{n+c+1} \le \sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n$$
$$\le 1$$

by Theorem 1, it follows that $F_c(z)$ is in the class $\sum_p(A, B)$.

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