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ON NONLINEAR FILTERING PROBLEM FOR AN OBLIQUE REFLECTING BROWNIAN MOTION*

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1. Introduction

There are two different approaches to the nonlinear filtering problem. The first approach is the innovations approach combined with representation theorems for continuous and discontinuous martingales as stochastic integral (see Fujisaki-Kallianpur-Kunita [1]). The second approach is focussed on the unnormalized conditional density equation, which is a stochastic partial differential equation so called Duncan-Mortensen-Zakai (for short, DMZ) equation (see Zakai [6]).

The aim of this paper is to derive DMZ equation corresponding to a Brownian motion with oblique reflecting boundary condition on an orthant.

In Section 2 we will formulate the problem and fix notations. In Section 3 we will give the proofs of our results. For general introduction to the nonlinear filtering problem theory see [3], [4] and their references.

2. Formulation of the problem

The problem we dicuss is as follows. Let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a reference family $(\mathcal{F}_t)_{t\geq 0}$. Let $D=\{x=(\xi, x_n)\in R^n \mid \xi\in \boldsymbol{R}^{n-1},\ x_n\in \boldsymbol{R}^1 \text{ and } x_n>0\}$, $D=\{x\in \boldsymbol{R}^n \mid x_n=0\}$ and $\bar{D}=D\cup \partial D$. Consider a process $X_t=(X_t^1, \cdots, X_t^n)$, as *the signal process*, defined by

$$\text{(2.1)}\qquad \begin{cases} X_t^i=x^i+B_t^i+\int_0^t \beta_i(\tilde{X}_s)\,d\phi_s, & i=1, 2, \cdots, n-1, \\ X_t^n=x^n+B_t^n+\phi_t \end{cases}$$

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where $B_t=(B_t^1, \cdots, B_t^n)$ is an n-dimensional $\mathcal{F}_t$-adapted Brownian motion, $X_t=(\tilde{X}_t, X_t^n)$ and ϕ_t is the local time of X_t^n at 0, i.e., ϕ_t is continuous $\mathcal{F}_t$-adapted process satisfying $\phi_0=0$ and

$$\phi_t=\int_0^t I_{\{\partial D\}}(X_s)\,d\phi_s=\int_0^t I_{\{0\}}(X_s^n)\,d\phi_s \tag{2.2}$$

for all $t\in[0, \infty)$ and β_i, $i=1, 2, \cdots, n-1$, are bounded measurable function on ∂D with bounded derivative of first order. Here I_A is the indicate function of the set A. This process X_t is called *an oblique reflecting Brownian motion* on the orthant $\bar{D}$. This process is one of a few examples of processes corresponding to non-symmetric Dirichlet spaces (see J. H. Kim [2]). And consider a d-dimensional process Y_t, as *the observation process* of X_t, defined by

$$dY_t=h(X_t)\,dt+dW_t \tag{2.3}$$

where W_t is a d-dimensional Brownian motion independent with B_t and $h: \mathbf{R}^n\to\mathbf{R}^d$ is bounded measurable. Let $\mathcal{G}_t$ be the σ-field generated by the observation $\{Y_t|0\leqq s\leqq t\}$ up to time t. The goal of nonlinear filtering problem theory is to study the conditional expection

$$\pi_t(f)=E[f(X_t)\,|\,\mathcal{G}_t]$$

taken with respect to the probability P, for suitable real valued function f. This is because $\pi_t(f)$ is the best estimate, in quadratic mean sense, of $f(X_t)$ given the observations $\mathcal{G}_t$. This estimate depends, in general, nonlinearly on the observations, and it is called *the nonlinear filter.*

3. DMZ equation for an oblique reflecting Brownian motion

In this section, we derive DMZ equation corresponding to the filtering problem by the signal process X_t defined by (2.1) and the observation process Y_t defined by (2.3).

LEMMA 3.1. *The differential operator* $(L, \mathcal{D}(L))$ *corresponding to the process defined by* (2.1) *is given by*

$$\begin{cases} L=\dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\dfrac{\partial^2}{\partial x_i^2}\ (\textit{in the sense of Schwartz distribution}), \\ \mathcal{D}(L)=\{u\in C_0^2(\bar{D})\dfrac{\partial u}{\partial x_n}+\displaystyle\sum_{i=1}^{n-1}\beta i\dfrac{\partial u}{\partial x_i}=0 \ \textit{on}\ \partial D\}. \end{cases} \tag{3.1}$$

Proof. By Ito's formula, for any $f\in C_0^2(\bar{D})$, we have

$$f(X_t)-f(X_0)=\sum_{i=1}^{n}\int_0^t \frac{\partial f}{\partial x_i}(X_s)\,dB_s^i+\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t \frac{\partial^2 f}{\partial x_i^2}(X_s)\,ds$$
$$+\sum_{i=1}^{n-1}\int_0^t \beta_i(X_s)\frac{\partial f}{\partial x_i}(X_s)\,d\phi_s+\int_0^t \frac{\partial f}{\partial x_n}(X_s)\,d\phi_s.$$

Thus, for any $f\in C_0^2(\bar{D})$ such that

$$L_0 f=\frac{\partial f}{\partial x_n}+\sum_{i=1}^{n-1}\beta_i\frac{\partial f}{\partial x_i}=0 \text{ on } \partial D,$$

we have

$$Lf(x)=\lim_{t\to 0}\frac{1}{t}E[f(X_t)-f(X_0)\,|\,X_0=x]=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i^2}(x).$$

The proof is complete.

For $f\in\mathcal{D}(L)$, define

$$\rho_t(f)=\pi_t(f)\,\alpha_t \tag{3.2}$$

where

$$\alpha_t=\exp\left\{\int_0^t \pi_s(h)\,dY_s-\frac{1}{2}\int_0^t|\pi_s(h)|^2 ds\right\}. \tag{3.3}$$

Using Lemma 3.1 and the same way as Theorem B and C in [3], we have the following theorem.

THEOREM 3.2. *Let $(L,\ \mathcal{D}(L))$ be the differential operator defined by* (3.1). *Then ρ_t defined by* (3.2) *is a solution of the following stochastic partial differential equation which is called a Zakai equation.*

$$\rho_t(f)=\rho_0(f)+\int_0^t\rho_s(Lf)\,ds+\int_0^t\rho_s(hf)\,dY_s.$$

Now we define $\sigma(t,x)$ on $[0,\infty)\times D$ by

$$\rho_t(f)=\int_D f(x)\sigma(t,x)\,dx \tag{3.4}$$

$\sigma(t,x)$ is called an unnormalized conditional density of X_t given $\mathcal{G}_t$ on D. Since $\{s\,|\,X_s\in\partial D\}$ has Lebesgue measure zero, unnormalized conditional density on ∂D is zero.

THEOREM 3.3. *The unnormalized conditional density $\sigma(t,x)$ defined by* (3.4) *is a solution of the following stochastic partial differential equation which is called a DMZ equation*:

$$\begin{cases} d\sigma(t,x)=L\sigma(t,x)\,dt+h\sigma(t,x)\,dY_t \ \text{ on } D\\ L_0\sigma(t,x)=0 \ \text{ on } \partial D.\end{cases} \tag{3.5}$$

Proof. For any $f\in\mathcal{D}(L)$, by Theorem 3.2,

$$\int_D f(x)\sigma(t, x)dx=\sigma_0(f)+\int_0^t \sigma_s(Lf)ds+\int_0^t \sigma_s(hf)dY_s$$
$$=\sigma_0(f)+\int_0^t\int_D Lf(x)\sigma(s,x)dxds+\int_0^t\int_D (hf)(x)\sigma(s,x)dxdY_s$$
$$=\sigma_0(f)+\int_0^t \frac{1}{2}\sum_{i=1}^n \int_D \frac{\partial^2 f(x)}{\partial x_i^2}\sigma(s,x)dxds+\int_0^t\int_D (fh)(x)\sigma(s,x)dxdY_s$$
$$=\sigma_0(f)+\int_0^t\Big[-\frac{1}{2}\sum_{i=1}^n \int_D \frac{\partial f(x)}{\partial x_i}\frac{\partial\sigma(s,x)}{\partial x_i}dx+\sum_{i=1}^n\int_{\partial D}\frac{\partial f(\xi,0)}{\partial x_i}$$
$$\sigma(s,(\xi,0))e_i d\xi\Big]ds+\int_0^t\int_D (fh)(x)\sigma(s,x)dxdY_s$$
$$=\sigma_0(f)+\int_0^t\Big[-\frac{1}{2}\sum_{i=1}^n \int_D \frac{\partial f(x)}{\partial x_i}\frac{\partial\sigma(s,x)}{\partial x_i}dx+\int_{\partial D}\frac{f(\xi,0)}{\partial x_n}\sigma(s,(\xi,0))d\xi\Big]ds$$
$$+\int_0^t\int_D (fh)(x)\sigma(s,x)dxdY_s$$
$$=\sigma_0(f)+\int_0^t\Big[\frac{1}{2}\sum_{i=1}^n \int_D f(x)\frac{\partial^2\sigma(s,x)}{\partial x_i^2}dx+\int_{\partial D}\frac{\partial f(\xi,0)}{\partial x_n}\sigma(s,(\xi,0))d\xi$$
$$+\sum_{i=1}^n\int_{\partial D} f(\xi,0)\frac{\partial\sigma(s,(\xi,0))}{\partial x_i}e_i d\xi\Big]ds+\int_0^t\int_D f(x)h(x)\sigma(s,x)dxdY_s$$
$$=\sigma_0(f)+\int_0^t\Big[\int_D f(x)L\sigma(s,x)dx+\int_{\partial D}\frac{\partial f(\xi,0)}{\partial x_n}\sigma(s,(\xi,0))d\xi$$
$$+\int_{\partial D} f(\xi,0)\frac{\partial\sigma(s,(\xi,0))}{\partial x_n}d\xi\Big]ds+\int_0^t\int_D f(x)h(x)\sigma(s,x)dxdY_s,$$

where $e_i=(0,\cdots,0,\overset{i}{1},0,\cdots,0)$ is n-dimensional unit vector. Hence

$$\int_D f(x)d_t\sigma(t,x)dx=\int_D f(x)L\sigma(t,x)dx+\int_{\partial D}\frac{\partial f(\xi,0)}{\partial x_n}\sigma(t,(\xi,0))d\xi$$
$$+\int_D f(x)h(x)\sigma(t,x)dY_t dx+\int_{\partial D} f(\xi,0)\frac{\partial\sigma(t,(\xi,0))}{\partial x_n}d\xi.$$

From this, we have $d\sigma(t,x)=L\sigma(t,x)+h(x)\sigma(t,x)dY_t$ and

$$\sigma(t,(\xi,0))=\frac{\partial\sigma(t,(\xi,0))}{\partial x_n}=0. \tag{3.6}$$

By (3.6) and the same argument as above, we have

$$\sum_{i=1}^{n-1}\beta_i(\xi)\frac{\partial\sigma(t,(\xi,0))}{\partial x_i}=0.$$

Thus $L_0\sigma(t,x)=0$ on ∂D. The proof is complete.

The following remark is due to J. H. Kim [2, Section 6]. This result will be used to establish the uniqueness of solution of stochastic partial differential equation (3.5).

REMARK 3.4. *We define the Sobolev space*

$$H^1(D)=\{u\in L^2(D)\,|\,\frac{\partial u}{\partial x_i}\in L^2(D),\quad i=1,2,\cdots,n\}.$$

Equipped with the norm

$$|u|_{H^1(D)}=|u|_{L^2(D)}+|u_x|_{L^2(D)}$$

where

$$|u_x|_{L^2(D)}=\left(\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2_{L^2(D)}\right)^{\frac{1}{2}}.$$

Let $a(.,.)$ *be the bilinear form corresponding to* L *defined by* (3.1), *i.e.*,

$$a(u,v)=(-2Lu,v)_{L^2(D)},\quad u,v\in H^1(D).$$

Then we have, for $u,v\in H^1(D)$,

$$a(u,v)=\sum_{i=1}^{n}\int_D\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx-\sum_{i=1}^{n-1}\int_{\partial D}\beta_i(\xi)v(\xi,0)\frac{\partial u(\xi,0)}{\partial\xi_i}d\xi$$

and this is a nonsymmetric Dirichlet form on $\bar{D}$. *For some* $\alpha_0>0$ *and any* $\alpha>\alpha_0$, *there exists a constant* $K=K(\alpha)>0$ *such that*

$$a(u,v)+\alpha(u,v)_{L^2(D)}\geqq K_1|u|^2_{H^1(D)}$$

for every $u\in H^1(D)$.

Now we establish the uniqueness of the solution of (3.5).

THEOREM 3.5. *The unnormalized conditional density* $\sigma(t,x)$ *defined by* (3.4) *is the unique solution of* (3.5) *with an initial condition* $\sigma(0,x)=\sigma_0(x)\in H^1(D)$.

Proof. We define a new probability $\bar{P}$, which is equivalent to P on each $\mathcal{F}_t$, by

$$\left.\frac{d\bar{P}}{dP}\right|_{\mathcal{F}_t}=\exp\{-\int_0^t h(X_s)dY_s-\frac{1}{2}\int_0^t|h(X_s)|^2ds\}.$$

Then, under $\bar{P}$, the observation process Y_t is a Brownian motion (see Lemma 2.1 in [4]). And, by Remark 3.4, we have

$$2(-Lu,u)_{L^2(D)}+(\alpha+\sum_{k=1}^{d}|h_k|^2_{L^2(D)})|u|^2_{L^2(D)}\geqq K|u|_{H^1(D)}+\sum_{k=1}^{d}|h_ku|^2_{L^2(D)}$$

for any $u\in M^2(0,T;H^1(D))$, where

$M^2(0,T;H^1(D))=\{u\in L^2((0,T)\times\Omega\to H^1(D))\,|\,u(t)$ is $\mathcal{F}_t$-adapted a.e. in $(0,T)\}$.

Thus, by the same way as Theorem 2.3 of Chapter II in *E*. Pardoux [5], the equation (3.5) with above initial condition has the unique solution. From Theorem 3.4, we see that $\sigma(t,x)$ is the unique solution of (3.5). The proof is complete.

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