J. Korean Math. Soc. 27(1990), No. 2, pp. 231-237

KAEHLERIAN MANIFOLDS WITH A QUASI-UMBILICAL HYPERSURFACE AT EACH POINT

IN-BAE KIM AND JONG TAEK CHO

1. Introduction

It is well-known that the axiom of n-planes by E. Cartan ([2]) and that of n-spheres by D. S. Leung and K. Nomizu ([7]) give a characterization of spaces of constant curvature. Recently N. Innami and K. Shiga ([4]) introduced the axiom of generalized hyperspheres and applied it to geodesic spheres and horospheres to investigate spaces of constant curvature.

During the last year the study of real hypersurfaces of Kaehlerian manifolds has been an important subject in geometry of submanifolds ([3], [5], [9] and [11]). A Kaehlerian manifold of constant holomorphic sectional curvature is said to be a *complex space form.* Especially many authors have concerned themselves with real hypersurfaces of complex space forms ([6], [8] and [9]).

Let (M, J) be a real $2n$-dimensional Kaehlerian manifold, where J is the complex structure on M, and let $N(p)$ be a hypersurface through each point p of M. For a unit normal vector field V of N in M, the vector field

(1.1) $$\xi = JV$$

is tangent to N. If the shape operator $A : TN \to TN$ is expressed by

(1.2) $$AX = \alpha X + \beta\eta(X)\xi$$

for any vector field X on N and some scalar fields α and β on M, then N is said to be *quasi-umbilical,* where η is the associated 1-form of ξ. This terminology "quasi-umbilical" is the extended one of previous works (for instance, see [3], [6] and [10]). A geodesic through p on M generated by the unit normal vector field V of N will be

Received December 6, 1989.
Revised March 16, 1990.
This work was supported by KOSEF.

denoted by $C_p(s)$, where s is the arc-length of $C_p(s)$.

In terms of local coordinates (x^λ) of M and (u^h) of N, let

(1.3) $$N_t=\{p\in M\,|\,x^\lambda=x^\lambda(u^h, s),\ \ s=t \text{ at } p\}$$

for a fixed t, where the indices $\kappa, \lambda, \mu, \cdots$ and $h, i, j, \cdots$ run over the ranges $\{1, 2, \cdots, 2n\}$ and $\{1, 2, \cdots, 2n-1\}$ respectively. Then N_s is a hypersurface in M. If M satisfies the following; *for each* $p\in M$ *and every* $(2n-1)$*-dimensional subspace* T_p' *of the tangent space* $T_p(M)$ *of* M*, there exists a hypersurface* N *through* p *such that* $T_p(N)=T_p'$ *and* N_s *is quasi-umbilical at* $C_p(s)$ *for each* s, then we say that M is a *Kaehlerian manifold with a quasi-umbilical hypersurface at each point.*

Throughout this paper we assume that manifolds and quantities are differentiable and of class C^∞. The main purpose of this paper is to prove

THEOREM. *Let* M *be a real* $2n$ $(n\geq 2)$*-dimesnional Kaehlerian manifold with a quasi-umbilical hypersurface at each point. Then* M *is a complex space form with constant holomorphic sectional curvature* k*. Moreover the scalar fields* α *and* β *on* M *in* (1.2) *are given by*

$$\alpha=c\tan c\ (s+a),\quad \beta=-c\cot c\ (s+a)$$

provided $k=c^2>0$*, and*

$$\alpha=-c\tanh c\ (s+b),\quad \beta=-c\coth c\ (s+b)$$

provided $k=-c^2<0$*, where* a *and* b *are constants.*

2. Preliminaries

Let M be a real $2n$-dimensional Kaehlerian manifold with the structure (G, J), where G is the Kaehlerian metric tensor and J the complex structure tensor. The structure (G, J) satisfies

(2.1) $$J^2=-I,$$

I being the identity tensor field of M,

(2.2) $$G(JX, JY)=G(X, Y)$$

and

(2.3) $$\tilde{\nabla}_X J=0$$

for any vector fields X and Y on M, where $\tilde{\nabla}$ is the Riemannian connection with respect to the metric G. Let R be the Riemannian curvature tensor of M. Then it is known that

(2.4) $$R(JX, JY)Z=R(X, Y)Z$$

for any vector fields X, Y and Z on M.

Let N be a hypersurface through a point p in M and V a unit normal vector field of N in M. Then, for a local coordinate system (x^λ) on M, there is a coordinate system $(x^\lambda)=(u^h, s)$ of M about p, which is called an *adapted coordinate system*, such that (u^h) is a local coordinate system of N and s is the arc-length of a geodesic $C_p(s)$ generated by V. Therefore we can consider a transformation of the two coordinates on M such as

(2. 5) $$x^\lambda=x^\lambda(u^h, s).$$

If we put

(2. 6) $$B_i{}^\kappa=\partial_i x^\kappa, \quad \partial_i=\frac{\partial}{\partial u^i},$$

then $B_i=(B_i{}^\kappa)$ are $(2n-1)$ local vector fields on N spanning the tangent space $T_p(N)$ at every point p in N. Therefore the vectors B_i and V span the tangent space $T_p(M)$ of M at every point p in N, and the matrix

$$B=(B_i{}^\kappa, \ V^\kappa),$$

is regular. A Riemannian metric tensor $g=(g_{ij})$ of N is naturally induced from G of M as

(2. 7) $$g_{ji}=G_{\lambda\kappa}B_j{}^\lambda B_i{}^\kappa,$$

where summation convention is applied to repeated indices on their own ranges. Then the inverse matrix B^{-1} of B is given by

$$B^{-1}=\begin{bmatrix} B_\lambda{}^h \\ V_\lambda \end{bmatrix},$$

where $B_\lambda{}^h=g^{ik}G_{\lambda\kappa}B_i{}^\kappa$ and $V_\lambda=G_{\lambda\kappa}V^\kappa$.

The *van der Waerden–Bortolotti covariant differentiation* ∇_j is defined by

(2. 8) $$\nabla_j B_i{}^\kappa=\partial_j B_i{}^\kappa+\Gamma^\kappa_{\mu\lambda}B_j{}^\mu B_i{}^\lambda-B_h{}^\kappa\left\{ {h \atop j \ i} \right\},$$

where $\Gamma^\kappa_{\mu\lambda}$ and $\left\{ {h \atop j \ i} \right\}$ are the Christoffel symbols of M and N with respect to their metrices G and g respectively. The connection ∇_j and the Riemannian connection $\tilde{\nabla}$ on M are related to

$$\nabla_j=B_j{}^\lambda\tilde{\nabla}_\lambda.$$

Since $\nabla_j B_i{}^\kappa$ is normal to N for fixed i and j, we have the Gauss formula

(2. 9) $$\nabla_j B_i{}^\kappa=H_{ji}V^\kappa,$$

where $A=(H_j{}^i)$ is the shape operator or the second fundamental tensor of N in M and $H_{ji}=g_{hi}H_j{}^h$. The Weingarten formula is given by

(2.10) $$\nabla_j V^\kappa = -H_j{}^i B_i{}^\kappa,$$

where we have put

$$\nabla_j V^\kappa = \partial_j V^\kappa + \Gamma^\kappa_{\mu\lambda} B_j{}^\mu V^\lambda.$$

The equation of Codazzi is given by

(2.11) $$\nabla_k H_{ji} - \nabla_j H_{ki} = B_k{}^\nu B_j{}^\mu B_i{}^\lambda V^\kappa R_{\nu\mu\lambda\kappa},$$

where $R_{\nu\mu\lambda}{}^\kappa = G^{\alpha\kappa} R_{\nu\mu\lambda\alpha}$ are components of the Riemannian curvature tensor R of M.

Since V is a unit normal vector field of N in M, then ξ defined by (1.1) is a unit tangent vector field of N and satisfies

$$\eta(\xi)=1 \text{ or } \xi^i\eta_i=1.$$

In terms of the adapted coordinate system $(x^\lambda)=(u^h, s)$ of M, components ξ^λ of ξ as the vector field on M and ξ^h of ξ as the vector field on N are expressed by

$$\xi^\kappa = \xi^h B_h{}^\kappa = J_\lambda{}^\kappa V^\lambda \text{ and } \xi^h = J_\lambda{}^\kappa V^\lambda B_\kappa{}^h.$$

Therefore the components of the associated 1-form η are given by

(2.12) $$\eta_i = J_{\lambda\kappa} V^\lambda B_i{}^\kappa.$$

3. Proof of Theorem

Since B_i's are $(2n-1)$ linearly independent vectors in $T_p(M)$, $p\in M$, then the distribution T_p' spanned by the B_i's at p is a $(2n-1)$-dimensional subspace of $T_p(M)$. Therefore, by the assumption, there exists a hypersurface N through p of M such that $T_p(N)=T_p'$. Let U be a normal neighborhood of p in N. For each $q\in U$, let V_q be the normal vector at q to N which is parallel to V with respect to the normal connection along the geodesic $C_p(s)$ from p to q in U. Along each geodesic we have

(3.1) $$V^\lambda \tilde{\nabla}_\lambda V^\kappa = 0.$$

Differentiating (3.1) covariantly along N, we also obtain

(3.2) $$B_j{}^\nu(\tilde{\nabla}_\nu V^\lambda)(\tilde{\nabla}_\lambda V^\kappa) + V^\lambda B_j{}^\nu \tilde{\nabla}_\nu \tilde{\nabla}_\lambda V^\kappa = 0.$$

Since the components V^κ of V are given by

$$V^\kappa = \partial x^\kappa/\partial s$$

for the adapted coordinate system $(x^\lambda)=(u^h, s)$ and the Christoffel symbols $\Gamma^\kappa_{\mu\lambda}$ are symmetric for μ and λ, then it is easily seen that the covariant derivative of B_i's with respect to V and that of V with respect to B_i's are related to

(3.3) $$V^\lambda\tilde{\nabla}_\lambda B_i^\kappa=\partial_s B_i^\kappa+\Gamma_{\mu\lambda}^\kappa V^\mu B_i^\lambda=B_i^\lambda\tilde{\nabla}_\lambda V^\kappa.$$

Computing $V^\mu\tilde{\nabla}_\mu(V^\lambda\nabla_\lambda B_i^\kappa)$ and using (3.2) and (3.3), we have

$$V^\mu\nabla_\mu(V^\lambda\nabla_\lambda B_i^\kappa)=-B_j^\mu V^\lambda(\nabla_\mu\nabla_\lambda V^\kappa-\nabla_\lambda\nabla_\mu V^\kappa).$$

Therefore it follows from (2.9), (2.10) and the Ricci's identity that

(3.4) $$B_j^\nu R_{\nu\mu\lambda}{}^\kappa V^\mu V^\lambda=(\partial_s H_j{}^h-H_j{}^i H_i{}^h)B_h{}^\kappa,$$

where $R=(R_{\nu\mu\lambda}{}^\kappa)$ is the curvature tensor of M.

Since, for each s, N_s is a quasi-umbilical hypersurface by the assumption, it follows from (1.2) that components $H_j{}^i$ of the shape operator A of N_s are given by

(3.5) $$H_j{}^i=\alpha\delta_j^i+\beta\eta_j\xi^i,$$

where δ_j^i indicate components of the identity transformation on N. Substituting (3.5) into (3.4), we have

(3.6) $$B_j^\nu R_{\nu\mu\lambda}{}^\kappa V^\mu V^\lambda=[(\alpha_s-\alpha^2)\delta_\nu^\kappa g_{\mu\lambda}-(\beta_s-2\alpha\beta-\beta^2)J_{\nu\mu}J_\lambda{}^\kappa]V^\mu V^\lambda B_j^\nu$$

where $\alpha_s=\dfrac{\partial}{\partial s}\alpha$ and $\beta_s=\dfrac{\partial}{\partial s}\beta$. Since $BB^{-1}=I=(\delta_\lambda^\kappa)$, that is,

$$B_\lambda^i B_i^\kappa=\delta_\lambda^\kappa-V_\lambda V^\kappa,$$

then, applying B_ε^j to (3.6), we can find

(3.7)
$$R_{\nu\mu\lambda}{}^\kappa V^\mu V^\lambda=[(\alpha_s-\alpha^2)(\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda})-(\beta_s-2\alpha\beta-\beta^2)J_{\nu\mu}J_\lambda{}^\kappa]V^\mu V^\lambda.$$

If we put

(3.8) $$S_{\nu\mu\lambda}{}^\kappa=R_{\nu\mu\lambda}{}^\kappa-(\alpha_s-\alpha^2)(\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda})+(\beta_s-2\alpha\beta-\beta^2)J_{\nu\mu}J_\lambda{}^\kappa,$$

then the equation (3.7) shows that $S_{\nu\mu\lambda}{}^\kappa V^\mu V^\lambda=0$. Since V^λ is an arbitrary component of the vector field V, then it follows from (3.8) that

$$S_{\nu\mu\lambda}{}^\kappa+S_{\nu\lambda\mu}{}^\kappa=0,$$

Comparing this equation with (3.8), we have

(3.9) $$R_{\nu\mu\lambda}{}^\kappa+R_{\nu\lambda\mu}{}^\kappa=(\alpha_s-\alpha^2)(2\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda}-\delta_\lambda^\kappa g_{\nu\mu})$$
$$-(\beta_s-2\alpha\beta-\beta^2)(J_\mu{}^\kappa J_{\nu\lambda}+J_\lambda{}^\kappa J_{\nu\mu}).$$

By an argument similar to the above procedure, we also obtain

$$R_{\mu\nu\lambda}{}^\kappa+R_{\mu\lambda\nu}{}^\kappa=(\alpha_s-\alpha^2)(2\delta_\mu^\kappa g_{\nu\lambda}-\delta_\nu^\kappa g_{\mu\lambda}-\delta_\lambda^\kappa g_{\mu\nu})$$
$$-(\beta_s-2\alpha\beta-\beta^2)(J_\nu{}^\kappa J_{\mu\lambda}+J_\lambda{}^\kappa J_{\mu\nu}).$$

If we make the difference of (3.9) from the above and apply the first Bianchi's identity, then we have

(3.10) $$R_{\nu\mu\lambda}{}^\kappa=(\alpha_s-\alpha^2)(\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda})$$
$$+\frac{1}{3}(\beta_s-2\alpha\beta-\beta^2)(J_\nu{}^\kappa J_{\mu\lambda}-J_\mu{}^\kappa J_{\nu\lambda}-2J_\lambda{}^\kappa J_{\nu\mu}).$$

Applying $B_j^\nu\tilde{\nabla}_\nu=\nabla_j$ to (2.12) and using (2.9), (2.10), (3.5) and the relation $J_{\mu\lambda}\xi^h B_h^\lambda B_j^\mu=0$, we have

(3.11) $$\nabla_j\eta_i=-\alpha J_{\mu\lambda}B_j{}^\mu B_i{}^\lambda.$$

If we substitute (3.5) and (3.10) into the equation of Codazzi (2.11), then it follows from (2.12) and (3.11) that

(3.12) $$(\nabla_k\alpha)g_{ji}-(\nabla_j\alpha)g_{ki}+(\nabla_k\beta)\eta_j\eta_i-(\nabla_j\beta)\eta_k\eta_i$$
$$-\frac{1}{3}(\beta_s+\alpha\beta-\beta^2)J_{\mu\lambda}(2B_k{}^\mu B_j{}^\lambda\eta_i+B_k{}^\mu\eta_jB_i{}^\lambda-\eta_kB_j{}^\mu B_i{}^\lambda)$$
$$=0.$$

Applying ξ^i to (3.12), we obtain

(3.13) $$\eta_j\nabla_k(\alpha+\beta)-\eta_k\nabla_j(\alpha+\beta)-\frac{2}{3}(\beta_s+\alpha\beta-\beta^2)J_{\mu\lambda}B_k{}^\mu B_j{}^\lambda=0$$

and, applying ξ^j again to (3.13),

(3.14) $$\nabla_k(\alpha+\beta)=[\xi^h\nabla_h(\alpha+\beta)]\eta_k.$$

Thus it follows from (3.13) and (3.14) that

(3.15) $$\beta_s+\alpha\beta-\beta^2=0.$$

If we compare (2.4) with (3.10), then we can verify that

$$(\alpha_s-\alpha^2)(J_\omega{}^\kappa J_{\rho\lambda}-J_\rho{}^\kappa J_{\omega\lambda}-\delta_\omega{}^\kappa g_{\rho\lambda}+\delta_\rho{}^\kappa g_{\omega\lambda})$$
$$=\alpha\beta(\delta_\omega{}^\kappa g_{\rho\lambda}-\delta_\rho{}^\kappa g_{\omega\lambda}-J_\omega{}^\kappa J_{\rho\lambda}+J_\rho{}^\kappa J_{\omega\lambda}),$$

which is reduced to

(3.16) $$\alpha_s-\alpha^2=-\alpha\beta$$

by the contraction of $\omega=\kappa$. Therefore the comparison of (3.10) with (3.15) and (3.16) gives rise to the inquired equation

(3.17) $$R_{\nu\mu\lambda}{}^\kappa=(\alpha_s-\alpha^2)(\delta_\nu{}^\kappa g_{\mu\lambda}-\delta_\mu{}^\kappa g_{\nu\lambda}+J_\nu{}^\kappa J_{\mu\lambda}-J_\mu{}^\kappa J_{\nu\lambda}-2J_{\nu\mu}J_\lambda{}^\kappa).$$

If we make use of (3.17) and the second Bianchi's identity then we have

(3.18) $$(2n+1)g_{\mu\lambda}\tilde{\nabla}_\delta(\alpha_s-\alpha^2)-(2n+1)g_{\delta\lambda}\tilde{\nabla}_\mu(\alpha_s-\alpha^2)$$
$$+\tilde{\nabla}_\nu(\alpha_s-\alpha^2)(J_\mu{}^\nu J_{\delta\lambda}-J_\delta{}^\nu J_{\mu\lambda}-2J_{\mu\delta}J_\lambda{}^\nu)=0.$$

Applying $g^{\mu\lambda}$ to (3.18), we see from $n\geq 2$ that

$$\tilde{\nabla}_\lambda(\alpha_s-\alpha^2)=0,$$

which shows that $\alpha_s-\alpha^2$ is constant on M. Thus the equation (3.17) implies that M is a complex space form.

If we put $\alpha_s-\alpha^2=k$, k being a constant, then it is easily verify from (3.16) that

$$\alpha=c\tan c(s+a),\quad \beta=-c\cot c(s+a)$$

provided $k=c^2>0$, and

$$a=-c\tanh c(s+b),\quad \beta=-c\coth c(s+b)$$

provided $k=-c^2<0$, where a and b are constants. This completes the proof.

References

1. H. Busemann, The Geometry of Geodesics, Academic Press, New York, 1955.
2. E. Cartan, Lecons sur la Géométrie des Espaces de Riemann, Gauthier-Villars, Paris, 1946.
3. B.-Y. Chen and L. Vanheche, *Differential geometry of geodesic spheres,* J. Reine Angew Math. **325** (1981), 28-67.
4. N. Innami and K. Shiga, *The axiom of generalized hyperspheres in Riemannian geometry,* Kōdai Math. J., **8** (1985), 259-263.
5. I.-B. Kim and J.H. Kwon, *Submanifolds of Kaehlerian manifolds,* J. Korean Math. Soc., **23** (1986), 223-239.
6. M. Kon, *Pseudo-Einstein real hypersurfaces in complex space forms,* J. Diff. Geom., **14** (1979), 339-354.
7. D.S. Leung and K. Nomizu, *The axiom of spheres in Riemannian geometry,* J. Diff. Geom., **5** (1971), 487-489.
8. S. Montiel, *Real hypersurfaces of a complex hyperbolic space,* J. Math. Soc. Japan, **37** (1985), 515-535.
9. Y. Tashiro and S. Tachibana, *On Fubinian and C-Fubinian manifolds,* Kodai Math. Sem. Rep. **15** (1963), 176-183.
10. D. Van Lindt and L. Verstraelen, *A survey on axioms of submanifolds in Riemannian and Kaehlerian geometry,* Colloquium Math., **54**(1987), 193-213.
11. K. Yano and M. Kon, Structures on Manifolds, World Scientific, Singapore, 1984.

Hankuk University of Foreign Studies
Seoul 130-791, Korea