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CONTINUOUS AND LINEAR SELECTIONS FOR THE METRIC PROJECTION

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1. Introduction

A linear subspace M of a normed linear space X is called *proximinal* (resp. *Chebyshev*) if, for each x in X, the set of best approximations to x from M,

$$P_M(x) := \{y \in M : \|x-y\| = \inf_{m\in M} \|x-m\|\}, \qquad (1\text{-}1)$$

is nonempty (resp. a singleton). The set-valued mapping $P_M : X \to 2^M$ thus defined is called the metric projection onto M. A selection for P_M or a metric selection for M is a function $s : X \to M$ such that $s(x) \in P_M(x)$ for all $x \in X$. In this paper, we are mainly interested in selections which are also continuous or linear.

Let $H(M)$ denote the collection of all nonempty, closed, bounded and convex subsets of M. It is well-known that if M is proximinal, then $P_M : X \to H(M)$ and P_M is homogeneous, i.e., $P_M(\alpha x) = \alpha P_M(x)$ for all $x \in X$ and $\alpha \in \mathbf{R}$, and P_M is additive, i.e., $P_M(x+m) = P_M(x) + m$ for all $x \in X$ and $m \in M$.

A selections s for P_M is said to be:

$$\text{homogeneous if } s(\alpha x) = \alpha s(x),\ x \in X,\ \alpha \in \mathbf{R}, \qquad (1\text{-}2)$$

$$\text{additive if } s(x+m) = s(x) + m,\ x \in X,\ m \in M. \qquad (1\text{-}3)$$

Finally, the kernel of the metric projection P_M is the set

$$\text{Ker } P_M := \{x \in X : 0 \in P_M(x)\}.$$

Now we can outline some of the main results of this paper. In section 2, we give general results for continuous seleection from [5].

In Section 3, consider the space $L_1 = L_1(T, \mathcal{S}, \mu)$ of the functions on the measure space $(T, \mathcal{S}, \mu)$. For an n-dimensional subspace M of L_1 which has a basis $\{y_1, y_2, \cdots, y_n\}$ satisfying $\mu\ \{\text{supp}(y_i) \cap \text{supp}(y_j)\} = 0$

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for all $i \neq j$, we proved the following result. P_M has a continuous selection if and only if each x_j satisfies the Lazar condition.

In Seciton 4, we study the linear metric projection on $L_p = L_p(T, \mathcal{S}, \mu)$. By using the results from [7], for an n-dimensional subspace M of L_p we proved the following result. P_M is linear if and only if there exists a basis $\{m_1, m_2, \cdots, m_n\}$ of M such that $\mathrm{supp}(m_i)$ is purely atomic and contains at most two atoms.

In Section 5, for an n-dimensional subspace M of L_1 we prove the following theorem by using results in [1] and [7]. P_M has a linear selection if and only if there exists a basis $\{m_1, m_2, \cdots, m_n\}$ of M such that $\mathrm{supp}(m_i)$ contains an atom A_i for $i=1, 2, \cdots, n$ so that

$$\|m_i\| \leqq 2|m_i(A_i)\mu(A_i)| \text{ and } m_i(A_j)=0 \text{ for } i \neq j.$$

In Section 6, we learn something from the proof of Theorem 11 in [7].

2. General theorem

A subset N of X is called homogeneous if $\alpha N \subset N$ for each $\alpha \in \boldsymbol{R}$. If M is a proximinal subspace of X and N is a subset (not necessarily a subspace) of X, we will write $X = M \oplus N$ to mean that each $x \in X$ has a unique representation as $x = m + n$, where $m \in M$ and $n \in N$.

Recall that the quotient map $Q = Q_M : X \to X/M$ defined by $Q(x) = x + M$, is linear, $\|Q(x)\| \leqq \|x\|$ for every x, and $\|Qx\| = \|x\|$ for each $x \in \mathrm{Ker} P_M$.

We can now characterize when the metric projection has a continuous selection which is additive modulo M.

THEOREM 2.1. [5] *Let M be a proximinal subspace of a Banach space X. The following are equivalent*:

(1) *P_M has a continuous selection*;

(2) *P_M has a continuous selection which is homogeneous and additive modulo M*;

(3) *Ker P_M contains a closed homogeneous subset N such that $X = M \oplus N$ and the mapping $s(m+n) = m$ is continuous*;

(4) *KerP_M contains a closed homogeneous subset N such that $Q|_N$ is a homeomorphism between N and X/M.*

COROLLARY 2.2. [5] *Let M be a Chebyshev subspace of a normed space*

X. The following statements are equivalent:

(1) P_M *is continuous;*

(2) $(Q|_{\mathrm{Ker}P_M})-1$ *is continuous.*

REMARKS. (1) In [1], we can find characterization of subspaces of a normed linear space whose metric projection has a linear selection.

(2) We can find more informations of general results of continuous, and Lipschitz continuous selections in [5].

3. Continuous selections in L_1

Let $(T, \mathcal{S}, \mu)$ be a measure space and $L_1=L_1(T, \mathcal{S}, \mu)$ denote the space of all real-valued measurable functions x on T which are integrable and having the norm

$$\|x\| := \int_T |x(t)|\, d\mu.$$

An *atom* is a set $A\in\mathcal{S}$ such that $0<\mu(A)<\infty$ and if $B\in\mathcal{S}$, $B\subset A$, then either $\mu(B)=0$ or $\mu(B)=\mu(A)$. A measurable subset E of T is called *purely atomic* if E is (up to a set of measure zero) a union of atoms. Any measurable function x is constant a. e. (μ) on an atom A. We will write $x(A)$ for this value. For $x\in L_1$, the support of x and zero set of x are defined (up to a set of measure zero) by $\mathrm{supp}x=\{t\in T : x(t)\neq 0\}$ and $Z(x)=T\setminus\mathrm{supp}x=\{t\in T : x(t)=0\}$.

DEFINITION 3. 1. [3] Let $y\in L_1(T)$. We say that y satisfies the *Lazar condition* if whenever A and B are disjoint measurable sets with $A\cup B=\mathrm{supp}y$ and

$$\int_A |y|\, d\mu=\int_B |y|\, d\mu,$$

then either A or B must be a finite union of atoms.

THEOREM 3. 2. [3] *Let* $y\in L_1\setminus\{0\}$. *Then* $P_{[y]}$ *has a continuous selection if and only if* y *satisfies the Lazar condition.*

THEOREM 3. 3. *Let* $y_1, y_2, \cdots, y_n\in L_1\setminus\{0\}$ *be such that* $\mu(supp(y_i)\cap supp(y_j))=0$ *if* $i\neq j$. *Then* $P_{[y_1, y_2, \cdots, y_n]}$ *has a continuous selection if and only if* $y_1, y_2, \cdots y_n$ *satisfy the Lazar condition.*

Proof. We proceed by induction.

(i) $n=1$. By Theorem 3. 2, it holds.

(ii) Assume that $P_{[y_1,y_2,\cdots,y_k]}$ has a continuous selection if and only if $y_1, y_2, \cdots, y_k$ satisfy the Lazar condition.

(iii) We need to prove it for $n=k+1$.

Assume $P_{[y_1,y_2,\cdots,y_{k+1}]}$ has a continuous selection, say s. Then for each $x\in L_1(T)$, $s(x)=\alpha_1^x y_1+\alpha_2^x y_2+\cdots+\alpha_k^x y_k+\alpha_{k+1}^x y_{k+1}$. Define $s_1 : L_1 \to[y_1, y_2, \cdots y_k]$ and $s_2 : L_1\to[y_{k+1}]$ by

$$s_1(x)=\alpha_1^x y_1+\alpha_2^x y_2+\cdots+\alpha_k^x y_k \text{ and } s_2(x)=\alpha_{k+1}^x y_{k+1}.$$

Then $s(x)=s_1(x)+s_2(x)$. Let $x_n\to x$. Since s is continuous, $\|s_1(x_n)-s_1(x)\|_1+\|s_2(x_n)-s_2(x)\|_1=\|s(x_n)-s(x)\|_1\to 0$. Thus $\|s_1(x_n)-s_1(x)\|_1\to 0$ and $\|s_2(x_n)-s_2(x)\|_1\to 0$. So s_1 and s_2 are continuous.

Claim : s_1 is a selection for

$$P_{[y_1,y_2,\cdots,y_k]} : L_1|_{T-\mathrm{supp}(y_{k+1})}\to[y_1, y_2, \cdots, y_k].$$

Let $x\in L_1|_{T-\mathrm{supp}(y_{k+1})}=\{z\in L_1 : z(\mathrm{supp}(y_{k+1}))=0\}$. Then

$$\|x-s(x)\|_1=\int_{T-\mathrm{supp}(y_{k+1})}|x-s(x)|\,d\mu+\int_{\mathrm{supp}(y_{k+1})}|s_2(x)|\,d\mu$$

$$\leqq\int_{T-\mathrm{supp}(k+1)}|x-(\alpha_1y_1+\alpha_2y_2+\cdots+\alpha_ky_k)|\,d\mu+\int_{\mathrm{supp}(y_{k+1})}|\alpha_{k+1}y_{k+1}|\,d\mu$$

for any $\alpha_1, \alpha_2, \cdots, \alpha_{k+1}\in \boldsymbol{R}$. If $\alpha_{k+1}=0$, then

$$\int_{T-\mathrm{supp}(y_{k+1})}|x-s_1(x)|\,d\mu\leqq\int_{T-\mathrm{supp}(y_{k+1})}|x-\alpha_1y_1-\alpha_2y_2-\cdots-\alpha_ky_k|\,d\mu$$

for any $\alpha_1, \alpha_2, \cdots, \alpha_k\in \boldsymbol{R}$. Thus for each $x\in L_1|_{T-\mathrm{supp}(y_{k+1})}$, $s_1(x)\in P_{[y_1,y_2,\cdots,y_k]}(x)$. By assumption, $y_1, y_2, \cdots, y_k$ satisfy the Lazar condition. Similarly we can prove that $s_2 : L_1|_{T-\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}\to[y_{k+1}]$ is a selection for $P_{[y_{k+1}]} : L_1|_{T-\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}\to 2^{[y_{k+1}]}$. By theorem 3. 2, y_{k+1} satisfies the Lazar condition.

Conversely, assume that $y_1, y_2, \cdots, y_{k+1}$ satisfy the Lazar condition. By (ii), $P_{[y_1,y_2,\cdots,y_k]}$ has a continuous selection, say s_1. By Theorem 3. 2, $P_{[y_{k+1}]}$ has a continuous selection, say s_2.

Claim : s_1+s_2 is a continuous selection for $P_{[y_1,y_2,\cdots,y_k,y_{k+1}]}$. Since s_1 and s_2 are continuous, s_1+s_2 is continuous. It suffices to prove that, for each $x\in L_1$,

$$s_1(x)+s_2(x)\in P_{[y_1,y_2,\cdots,y_k,y_{k+1}]}(x).$$

Since $s_1(x)\in P_{[y_1,y_2,\cdots,y_k]}(x)$,

$$\|x-s_1(x)\|_1=\int_{T-\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}|x|\,d\mu+\int_{\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}|x-s_1(x)|\,d\mu$$

$$\leqq\|x-\alpha_1y_1-\alpha_2y_2-\cdots-\alpha_ky_k\|_1$$

$$=\int_{T-\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}|x|\,d\mu+\int_{\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}|x-\alpha_1y_1-\alpha_2y_2-\cdots-\alpha_ky_k|\,d\mu$$

for any $\alpha_1, \alpha_2, \cdots, \alpha_k \in \boldsymbol{R}$. Thus

$$\int_{\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}|x-s_1(x)|\,d\mu\leqq\int_{\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}|x-\alpha_1y_1-\alpha_2y_2-\cdots-\alpha_ky_k|\,d\mu$$

for any $\alpha_1, \alpha_2, \cdots, \alpha_k \in \boldsymbol{R}$. Similarly,

$$\int_{\mathrm{supp}(y_{k+1})}|x-s_2(x)|\,d\mu\leqq\int_{\mathrm{supp}(y_{k+1})}|x-\alpha_{k+1}y_{k+1}|\,d\mu$$

for any $\alpha_{k+1} \in \boldsymbol{R}$. Then

$$\|x-s_1(x)-s_2(x)\|_1=\int_{\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}|x-s_1(x)|\,d\mu+\int_{\mathrm{supp}(y_{k+1})}|x-\alpha_{k+1}y_{k+1}|\,d\mu$$
$$+\int_{T-\bigcup_{i=1}^{k+1}\mathrm{supp}(y_i)}|x|\,d\mu$$
$$\leqq\int_{\bigcup_{i=1}^{k}\mathrm{supp}(y_i)}|x-\alpha_1y_1-\alpha_2y_2-\cdots-\alpha_ky_k|\,d\mu$$
$$+\int_{\mathrm{supp}(y_{k+1})}|x-\alpha_{k+1}y_{k+1}|\,d\mu+\int_{T-\bigcup_{i=1}^{k+1}\mathrm{supp}(y_i)}|x|\,d\mu$$
$$=\|x-\alpha_1y_1-\alpha_2y_2-\cdots-\alpha_{k+1}y_{k+1}\|_1$$

for any $\alpha_1, \alpha_2, \cdots, \alpha_{k+1} \in \boldsymbol{R}$. Thus

$$s_1(x)+s_2(x)\in P_{[y_1,y_2,\cdots,y_{k+1}]}(x).$$

COROLLARY 3.4. *Let $y_1, y_2, \cdots, y_n \in l_1\setminus\{0\}$ be such that $supp(y_i)\cap supp(y_j)=\phi$ if $i\neq j$. $P_{[y_1,y_2,\cdots,y_n]}$ has a continuous selection if and only if $y_1, y_2, \cdots, y_n$ satisfy the Lazar condition.*

Because of the assumption on the supports of a basis, the above theorem is not a complete characterization of those subspace having continuous metric selections. This leads us to the next question: Is there an intrinsic characterization of the n-dimensional ($n>1$) subspaces of L_1 which have a continuous metric selection?

4. Linear selections in L_p

Let $(T, \mathcal{S}, \mu)$ be a measure space, $1<p<\infty$, and let $L_p=L_p(T, \mathcal{S}, \mu)$ denote the space of all real-valued measurable functions x on T whose absolute p^{th} power are integrable and whose norm is

$$\|x\|:=\left[\int_T|x(t)|^p d\mu\right]^{1/p}.$$

F. Deutsch [1] gave the following Theorem that is an intrinsic characterization of those $x_1 \in L_p \setminus \{0\}$ such that $P_{[x_1]}$ is linear.

THEOREM 4.1. [1] *Let $x_1 \in L_p \setminus \{0\}$, $1<p<\infty$, and $p \neq 2$. The following statements are equivalent:*

(1) *$P_{[x1]}$ is linear;*

(2) *$supp(x_1)$ is purely atomic and contains at most two atoms.*

when $p=2$, L_p is a Hilbert space, so for any closed subspace M, P_M is linear. We will therefore only be interested in the case when $p \neq 2$, i.e., when L_p is not a Hilbert space.

Let T_0 denote the union of all atoms in $(T, \mathcal{S}, \mu)$.

THEOREM 4.2. *Suppose M is an n-dimensional subspace of L_p, $1<p<\infty$, $p \neq 2$. The following statements are equivalent:*

(1) *P_M is linear;*

(2) *There exist k disjoint subsets $B_1, B_2, \cdots, B_k$ of T_0 such that $M=\bigoplus_{i=1}^{k} M_i$, where M_i is either $L_p(B_i)$ or a hyperplane in $L_p(B_i)$;*

(3) *There exist $l+1$ distinct subsets $B_0, B_1, \cdots, B_l$ of T_0 such that $M=L_p(B_0) \oplus (\bigoplus_{i=1}^{l} M_i)$ where M_i is a hyperplane of $L_p(B_i)$;*

(4) *There exists a basis $\{m_1, m_2, \cdots, m_n\}$ of M such that supp (m_i) is purely atomic and contains at most two atoms.*

Proof. (1)$\Leftrightarrow$(2) is proved by P.-K. Lin [7].

(2) $\Rightarrow$(3) Suppose (2) holds. Let $\mathcal{B}$ be the collection of B_i which $M_i=L_p(B_i)$. Put $B_0=\bigcup_{B_i \in B} B_i$ and $l=k$-the number of sets in $\mathcal{B}$. Then $L_p(B_0)=\bigoplus_{B_i \in B} L_p(B_i)$. Thus $M=L_p(B_0) \oplus (\bigoplus_{i=1}^{l} M_i)$ where M_i is a hyperplane of $L_p(B_i)$.

(3) $\Rightarrow$ (4) Suppose (3) holds. If $M=L_p(B_0)$, then B_0 has n atoms, say $\{A_1, A_2, \cdots, A_n\}$. Thus $m_i=\chi_{A_i}$, $i=1, 2, \cdots, n$. We may assume $M \neq L_p(B_0)$. For $L_p(B_0)$ we can find a basis of $L_p(B_0)$ consisting of characteristic functions of each atom of B_0. It suffices to prove that for M_1 there is a basis such that the support of each element of it is purely atomic and contains at most two atoms. Since M_1 is finite dimensional subspace, we can find a basis of M_1 such that the support of each element of it is purely atomic because B_1 is purely atomic.

By the elementary, property, we can find a basis for M_1 such that the support of each element of it contain at most two atoms. Similarly, we can find a basis of M_i such that the support of each element of it is purely atomic and contains at most two atoms. Therefore (4) holds.

(4) $\Rightarrow$(2) is easy to prove.

REMARK. F. Deutsch [1] proved (1)$\Leftrightarrow$(4) when dim $M=1$. P.-K. Lin [7] proved (1) $\Leftrightarrow$ (2).

5. Linear selections in L_1

In this section, we give intrinsic characterizations of those finite-dimensional subspaces of L_1 whose metric projections admits linear selections.

LEMMA 5.1. *Let M be an n-dimensional subspace of $L_1(T)$. Suppose that P_M has a linear selection s and there exist an atom A and $m_0(\neq 0)\in M$ such that*

$$|m_0(A)\mu(A)|\geqq\int_{T-A}|m_0(t)|d\mu.$$

Then the metric projection $P_{M_1}: L_1(T-A)\to M_1$ has a linear selection where $M_1=\{m\in M: m(A)=0\}$.

Proof. Let s be a linear selection for P_M.

Define $s_1: L_1(T-A)\to M_1$ by

$$s_1(f)=s(f')-\frac{s(f')(A)}{m_0(A)}m_0 \text{ where } f'=\begin{cases} f & \text{on } T-A \\ 0 & \text{on } A. \end{cases}$$

Then s_1 is well-defined since $m_0(A)\neq 0$. Since $s(f')\in M$, $\frac{s(f')(A)}{m_0(A)}m_0\in M$ and $s_1(f)(A)=0$, $s_1(f)\in M_1$. Since s is linear, s_1 is linear. To show s_1 is a linear selection for P_{M_1}, we must show that $s_1(f)\in P_{M_1}(f)$ for each $f\in L_1(T-A)$. Let $f\in L_1(T-A)$. Then

$$\begin{aligned}\|f-s_1(f)\|_{T\setminus A}&=\|f-(s(f')-\frac{s(f')(A)}{m_0(A)}m_0)\|_{T\setminus A}\\ &\leqq\|f-s(f')\|_{T\setminus A}+\left|\frac{s(f')(A)}{m_0(A)}\right|\|m_0\|_{T\setminus A}\\ &=\|f-s(f')\|_{T\setminus A}+\left|\frac{s(f')(A)}{m_0(A)}\right|\int_{T\setminus A}|m_0|\,d\mu\\ &\leqq\|f-s(f')\|_{T\setminus A}+|s(f')(A)\mu(A)|\\ &=\|f'-s(f')\|_T\leqq\|f'-m\|_T\end{aligned}$$

for any m in M. Since M_1 is a subspace of M,

$$\|f-s_1(f)\|_{T\setminus A}\leqq\|f'-m\|_T$$
$$=\|f-m\|_{T\setminus A}$$

for any m in M_1. Thus $s_1(f)\in P_{M_1}(f)$. Therefore s_1 is a linear selection for P_M in $L_1(T\setminus A)$.

Frank Deutsch [1] proved the following theorem.

THEOREM 5.2. *Let* $x_1\in L_1\setminus\{0\}$. *The following statements are equivalent:*

(1) $P_{[x_1]}$ *has a linear selection;*

(2) $Supp(x_1)$ *contains an atom and* $\|x_1\|\leqq 2\ max\{|x_1(A)\mu(A)| : A \text{ is an atom}\}$;

(3) $Supp(x_1)$ *contains an atom* A_0 *such that* $\|x_1\|\leqq 2|x_1(A_0)|\mu(A_0)$.

In the following theorem, we find that for an n-dimensional subspace the result in [7] is an extension of Theorem 5.2.

THEOREM 5.3. *Suppose that* M *is an* n*-dimensional subspace of* L_1. *The following statements are equivalent:*

(1) P_M *admits a linear selection;*

(2) *There exists a subset* T_1 *of* T *which contains exactly* n *atoms such that for any* $m\in M$

$$\int_{T_1}|m(t)|d\mu\geqq\int_{T\setminus T_1}|m(t)|d\mu\ ;$$

(3) *There exists a basis* $\{m_1, m_2, \cdots, m_n\}$ *of* M *such that* $supp(m_i)$ *contains an atom* A_i *for* $i=1, 2, \cdots, n$ *so that*

$$\|m_i\|\leqq 2|m_i(A_i)\mu(A_i)| \text{ and } m_i(A_j)=0 \text{ for } i\neq j;$$

(4) *There exists a basis* $\{m_1, m_2, \cdots, m_n\}$ *of* M *and* $\{A_1, A_2, \cdots, A_n\}\subset T_0$ *such that* $P_{[m_i]}$ *has a linear selection for* $i=1, 2, \cdots, n$ *and* $m_i(A_j)=0$ *for* $i\neq j$.

Proof. (1)$\Rightarrow$(2) is proven by Pei-Kee Lin [7]. In his proof, there may have a gap. In fact, he proved that if P_M has a selection, then there exists an atom A_1 in T and $m_1\neq 0\in M$ such that

$$|m_1(A_1)\mu(A_1)|\geqq\int_{T\setminus A_1}|m_1(t)|d\mu.$$

By the previous lemma, $P_{M_1}: L_1(T\setminus\{A_1\})\to M_1$ has a linear selection where $M_1=\{m\in M|m(A_1)=0\}$. Since $m_1\neq 0$, we can find a basis

$\{m_1, m_2', \cdots, m_n'\}$ of M. Let $m_i'' = m_i' - \frac{m_i'(A_1)}{m_1(A_1)} m_1$ for $i=2, \cdots, n$. Then $\{m_1, m_2'', \cdots, m_n''\}$ is a basis of M. So we may assume M_1 has $n-1$ dimension. By the same argument, there exist an atom A_2 in T and $m_2 \neq 0 \in M_1$ such that

$$|m_2(A_2)\mu(A_2)| \geqq \int_{T\setminus(A_1\cup A_2)} |m_2(t)|\, d\mu.$$

Replace m_1 by $m_1 - \frac{m_1(A_2)}{m_2(A_2)} m_2$ if necessary. So we may assume $m_1(A_2) = 0$. Continuing this process up to n, there exists $m_1, m_2, \cdots, m_n \in M$ and $\{A_1, A_2, \cdots, A_n\} = T_1 \subset T_0$ such that $m_i(A_j) = 0$ if $i \neq j$ and

$$\begin{aligned} |m_i(A_i)\mu(A_i)| &\geqq \int_{T\setminus A_i} |m_i(t)|\, d\mu \\ &= \int_{T\setminus T_1} |m_i(t)|\, d\mu \qquad \text{(5–1)} \end{aligned}$$

for any $i=1, 2, \cdots, n$. By (5–1), $\|m_i\| \leqq 2|m_i(A_i)\mu(A_i)|$ for any $i=1, 2, \cdots, n$. Since $m_i(A_j)=0$ if $i \neq j$, $\{m_1, m_2, \cdots, m_n\}$ is a basis for M. Thus we proved (1)$\Rightarrow$(3).

(3) $\Rightarrow$(2) Suppose that (3) holds. Since $\{m_1, m_2, \cdots, m_n\}$ is a basis of M, for any $m \in M$, $m = a_1m_1 + a_2m_2 + \cdots + a_nm_n$. Then

$$\begin{aligned} \int_{T_1} |m(t)|\, d\mu &= \sum_{i=1}^{n} |a_i|\, |m_i(A_i)\mu(A_i)| \\ &\geqq \sum_{i=1}^{n} |a_i| \int_{T\setminus T_i} |m_i(t)|\, d\mu \\ &= \sum_{i=1}^{n} \int_{T\setminus T_1} |a_i m_i(t)|\, d\mu \\ &= \int_{T\setminus T_1} \sum_{i=1}^{n} |a_i m_i(t)|\, d\mu \\ &\geqq \int_{T\setminus T_1} \Big|\sum_{i=1}^{n} a_i m_i(t)\Big|\, d\mu \\ &= \int_{T\setminus T_1} |m|\, d\mu. \end{aligned}$$

Thus for any $m \in M$,

$$\int_{T_1} |m(t)|\, d\mu \geqq \int_{T\setminus T_1} |m|\, d\mu.$$

Therefore (3)$\Rightarrow$(2) is proved.

(2) $\Rightarrow$(1) is proven by Pei-Kee Lin [7].

(3) $\Leftrightarrow$(4) By Theorem 5. 2, it is obvious.

REMARK. F. Deutsch [1] proved (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) when dim $M=1$.

Pei-Kee Lin [7] proved (1) $\Leftrightarrow$ (2). In fact, he proved (1) $\Leftrightarrow$ (3).

6. Linear selections in $C_0(T)$

Let T be a locally compact Hausdorff space of all real-valued continuous functions x on T which "vanish at infinity" (i. e., $\{t\in T : |x(t)|\geqq\varepsilon\}$ is compact for every $\varepsilon>0$) and endowed with the uniform norm : $\|x\|=\sup\{|x(t)| : t\in T\}$. When T is actually compact, $C_0(T)$ reduces to the space $C(T)$ of all continuous functions on T.

A point $t\in T$ is called an isolated point if the set $\{t\}$ is open.

THEOREM 6.1. *Let T be an n-dimensional subspace of $C_0(T)$. The following statements are equivalent*:

(1) *P_M admits a linear selection.*;

(2) *There exists a basis $\{m_1, m_2, \cdots, m_n\}$ of M and n isolated points $\{t_1, t_2, \cdots, t_n\}$ uch that $m_i(t_j)=\delta_{ij}$ for $i, j=1, 2, \cdots, n$ and card$(supp(m_i))\leqq 2$ for $i=1, 2, \cdots, n$*;

(3) *There exists a basis $\{m_1, m_2, \cdots, m_n\}$ of M such that card $(supp(m_i))\leqq 2$ for $i=1, 2, \cdots, n$*;

(4) *Let T_0 be the union of all isolated points of T. Then there exist k disjoint subsets $B_1, B_2, \cdots, B_k$ of T_0 such that $M=\bigoplus_{i=1}^{k} M_i$ where M_i is either $C(B_i)$ or a hyperplane of $C(B_i)$ for $i=1, 2, \cdots, k$*;

(5) *Let T_0 be the union of all isolated points of T. Then there exist $l+1$ disjoint subsets $D_0, D_1, \cdots, D_l$ of T_0 such that $M=C(D_0)\oplus(\bigoplus_{i=1}^{l} M_i)$ where M_i is either $C(D_i)$ or a hyperplane of $C(D_i)$ for $i=1, 2, \cdots, l$*;

Proof. Pei-Kee Lin [7] proved (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2) Suppose that (3) holds. By the elementary property, we can find n isolated points $\{t_1, t_2, \cdots, t_n\}$ such that $m_i(t_j)=\delta_{ij}$ for $i, j=1, 2, \cdots, n$. Thus (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (4) In the proof of (3) $\Rightarrow$ (4), Pei-Kee Lin defined $i\sim j$ if $\mathrm{supp}m_i\cap\mathrm{supp}m_j\neq\phi$. He said "$\sim$" is an equivalence relation on $\{1, 2, \cdots, n\}$. But it is not true because it does not satisfy the transitive law. Let supp $m_i=\{s_1, s_2\}$, supp$m_j=\{s_2, t_1\}$ and supp $m_k=\{t_1, t_2\}$ with $s_1\neq t_2$. Then $i\sim j$ and $j\sim k$ but not $i\sim k$. But if we assume that (2) holds, then "$\sim$" is an equivalence relation on $\{1, 2, \cdots, n\}$. Then Pei-Kee Lin's proof works for (2) $\Rightarrow$ (4).

(4) $\Rightarrow$ (5) Suppose that (4) holds. Let $\mathcal{D}$ be the collection of B_i which $M_i = C(B_i)$. Put $D_0 = \bigcup_{B_i \in D} B_i$ and $l = k -$ the number of sets in $\mathcal{D}$. Then $C(D_0) = \bigoplus_{B_i \in D} C(B_i)$. Rearrange l distinct subsets which are not contained in $\mathcal{D}$ as $D_1, D_2, \cdots, D_l$. Thus $M = C(D_0) \oplus (\bigoplus_{i=1}^{l} M_i)$ where M_i is a hyperplane of $C(D_i)$.

(5) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1) is proved by Pei-Kee Lin [7].

REMARK. F. Deutsch [1] proved (1) $\Leftrightarrow$ (3) wheu $\dim M = 1$. P.-K. Lin proved (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

References

1. F. Deutsch, *Linear selections for the metric projection*, J. Functional Anal. **49** (3) (1982), 269-292.
2. F. Deutsch, *A survey of metrtc selections*, Contemporary Mathematics **18** (1983), 49-71.
3. F, Deutsch, V. Indumathi, and K. Schnatz, *Lower semicontinuity, almost lower semicontinuity, and continuous selections*, J. Approx. Theory **53** (1989), 266-294.
4. F. Deutsch, *An exposition of recent results on continuous metric selections, Numerical Methods of Approximation Theory*, Vol. **8** (L. Collatz, G. Meinardus and G. Nurnberger, ed.), 67-79.
5. F. Deutsch, W. Li, and S. H. Park, *Characterizations of continuous and Lipschitz continuous metric selections in normed linear spaces*, J. Approx. Theory **58** (1989), 297-314.
6. A. J. Lazar, *Spaces of affine continuous functions on simplexes*, Trans. Amer. Math. Soc. **134** (1986), 503-525.
7. P.-K. Lin, *Remarks on linear selections for the metric projection*, J. Approx. Theory **43** (1985), 64-74.

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