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THE ROLE OF THE GENERALIZED EVALUATION SUBGROUPS IN THE FIXED POINT THEORY

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1. Notation and terminology

We will use the notion of [4], [5] and [7]. We give a brief review of the basic notions. Let X be a compact connected ANR, and let $p:\tilde{X}\to X$ be its universal covering. Let π be the group of Deck transformations of $(\tilde{X}, p)$, identified with $\pi_1(X, x_0)$ as usual, where x_0 is a base point of X. Let $f: X\to X$ be a map. Letting $\tilde{f}:\tilde{X}\to\tilde{X}$ be a lifting of f, then every lifting of f can be uniquely written as $\alpha\circ\tilde{f}$, $\alpha\in\pi$.

Two liftings $\tilde{f}, \tilde{f}'$ of f are conjugate iff there is a $\gamma\in\pi$ such that $\tilde{f}'=\gamma\circ\tilde{f}\circ\gamma^{-1}$. Each conjugate class $[\alpha\circ\tilde{f}]$ of liftings determines a fixed point class pFix$(\alpha\circ\tilde{f})$ of f. Any two elements $x_0, x_1\in\Phi(f)=\{x\in X\,|\,f(x)=x\}$ belong to the same fixed point class F iff there is a path c from x_0 to x_1 such that $c\approx f\circ c$ (homotopy rel endpoints). The index of a fixed point class $F=\text{pFix}(\alpha\circ\tilde{f})$ is denoted by index(f, F). If the fixed point index index$(f, F)\neq 0$ then F is called essential fixed point class and index$(f, F)=0$, then F is called inessential. It is known that if index$(f, F)=0$ then we can remove the fixed points in F by a map g homotopic to f in many cases. The Nielsen number $N(f)$ of a map f is defined to be the number of essential fixed point classes of f.

Let n be a natural number. We write $\tilde{f}^{(n)}$ for an arbitrary lifting of the iterate $f^n: X\to X$ and $\tilde{f}^n$ for the iterate of lifting $\tilde{f}$ of f. It is obvious that pFix$(\tilde{f})\subset$ pFix$(\tilde{f}^n)$. We define pFix$(\tilde{f}^n)$ to be the fixed point class of f^n containing pFix$(\tilde{f})$. Thus each fixed point class(empty or not) of f is contained a unique fixed piont class of f^n.

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Every $\tilde{f}^{(n)}$ can be factored as $\tilde{f}_n\circ\cdots\circ\tilde{f}_2\circ\tilde{f}_1$, where $\tilde{f}_i$ are liftings of $f, i=1, 2, 3, \cdots, n$. It follows from the commutativity of the index that for any fixed point class $F^{(n)}=\mathrm{pFix}(\tilde{f}_n\circ\cdots\circ\tilde{f}_2\circ\tilde{f}_1)$ of f^n, we have $\mathrm{index}(f^n, F^{(n)})=\mathrm{index}(f^n, fF^{(n)})$ because $f\circ f^{n-1}=f^{n-1}\circ f$. Note that if $F^{(n)}$ contains a fixed point class of f, i.e. $F^{(n)}=\mathrm{pFix}(\tilde{f}^n)$ for some lifting $\tilde{f}$ of f, then we have $fF^{(n)}=F^{(n)}$.

The following theorem is well known [4], [5].

THE MOD p INDEX THEOREM. *Suppose X is a compact connected ANR, $f: X\to X$ is a map. Suppose $n=p^r$, p=prime. Let $F^{(n)}$ be a fixed point class of f^n such that $F^{(n)}=fF^{(n)}$. Then $\mathrm{index}(f^n, F^{(n)})=\sum_F \mathrm{index}(f, F) \mod p$, where the summation is over all fixed point classes F of f contained in $F^{(n)}$.*

The generalized evaluation subgroups $G_n{}^f(X, A, x_0)$ [7] are defined by $G_n{}^f(X, A, x_0)=\mathrm{Im}(w_\pi : \pi_n(X^A, f)\to\pi_n(X, x_0))$ where $w: X^A\to X$ is the evaluation map from the function space X^A with compact open topology and $f: A\to X$ is a pointed map. $G_n{}^f(X, A, x_0)$ consists of all elements $\alpha\in\pi_n(X, x_0)$ such that there is a map $F: A\times S^n\to X$ with $F|_{A\times *}=f$ and $[f|_{*\times S^n}]=\alpha$. This definition is a generalization of the evaluation subgroup $G_n(X, x_0)$ [2][7] and independent to the base point. The subgroup $J(f, x_0)=G_1{}^f(X, X, x_0)$ is also called Jiang subgroup and plays many important roles in the fixed point theory [1], [4].

2. Some properties of $G_1{}^f(X, A, x_0)$

Woo [6] has showed that $G_1{}^i(X, A, x_0)\subset Z(i_\pi\pi_1(A, x_0),\ \pi_1(X, x_0))$ where $Z(H, K)$ denote the centralizer of a subgroup H in K and $i: A\subset X$ the inclusion. Similary we can have

THEOREM 1. *Let $f: (A, a_0)\to(X, x_0)$ be a map. Then*
$$G_1{}^f(X, A, x_0)\subset Z(f_\pi\pi_1(A, a_0),\ \pi_1(X, x_0)).$$

Moreover Woo [6] has proved that if A is a connected aspherical polyhedron and $i: A\subset X$ is an inclusion, then $G_1{}^i(X, A, x_0)=Z(i_\pi\pi_1(A, x_0),\ \pi_1(X, x_0))$.

Using the same method, we can generalize this theorem.

THEOREM 2. *Let A be a connected aspherical polyhedron and $f:(A,a_0)\to(X,x_0)$ be a map. Then*

$$G_1^f(X,A,x_0)=Z(f_\pi\pi_1(A,a_0),\ \pi_1(X,x_0)).$$

COROLLARY 3. *If A is a connected aspherical polyhedron and $f:A\to X$ is the map with $f_\pi\pi_1(A,a_0)\subset Z(\pi_1(X,x_0))$, then $G_1^f(X,A,x_0)=\pi_1(X,x_0)$.*

THEOREM 4. *Let $f:(A,a_0)\to(X,x_0)$ be a map. Then*

$$G_1^i(X,f(A),x_0)\subset G_1^f(X,A,x_0).$$

Proof. If $\alpha\in G_1^i(X,f(A),x_0)$, there is a map $H:f(A)\times I\to X$ such that $H(x,0)=H(x,1)=x$ and $[H(x_0,\ )]=\alpha$. Define $K:A\times I\to X$ by $K=H(f\times 1)$. Then K is the required homotopy for $\alpha\in G_1^f(X,A,x_0)$.

COROLLARY 5. $G_1^i(X,f(X),x_0)\subset J(f,x_0)$.

The proof of the next two theorems is left to the reader.

THEOREM 6. *Let $x_0\in A\subset B\subset X$ and $f:X\to X$ be a map with $f(x_0)=x_0$.*

Then we have $G_1^{f|B}(X,B,x_0)\subset G_1^{f|A}(X,A,x_0)$.

THEOREM 7. *Let $f:X\to X$ be a pointed map. Then*

$$G_1^f(X,X,x_0)\subset G_1^{f^2}(X,X,x_0)\subset\cdots$$

Summing up the above results we have the followings.

THEOREM 8. *Let $f:X\to X$ be a selfmap with $f(x_0)=x_0$. Then we have*

$$\begin{array}{ccccccc}
 & & G_1^f(X,f(X),x_0) & & & & \\
 & \cup & & \cap & & & \\
 & & J(f,x_0) & \subset G_1^{f^2}(X,X,x_0) & \subset\cdots\subset & G_1^{f^n}(X,X,x_0) & \subset\cdots \\
 & & \cup & \cup & & \cup & \\
G_1^i(X,f(X),x_0) & \subset & G_1^i(X,f^2(X),x_0) & & \subset\cdots\subset & G_1^i(X,f^n(X),x_0) & \subset\cdots
\end{array}$$

3. The generalized evaluation subgroup and Jiang subgroup

We will drop the base point from the notation for the fundamental group and the generalized evaluation subgroups. And we will assume that all maps have the base point as fixed point.

THEOREM 9. *Let* $f: X \to X$ *be a selfmap. Suppose* $g: f(X) \to f(X)$ *be a map such that* $fg = 1_{f(X)}$. *Then* $G_1^i(X, f(X)) = J(f) = G_1^f(X, f(X))$. *Moreover if* f *has right inverse* g, *then* $J(f) = J(X) = G_1^i(X, f(X))$, *where* $J(X) = J(1_X)$.

Proof. Let $H: f(X) \times I \to X$ be an associated map of $\alpha \in G_1^f(X, f(X))$. That is, $H(x', 0) = H(x', 1) = f(x')$ and $[H(x_0, \ )] = \alpha$. Then the map defined by $K = H(g \times 1)$ shows that $\alpha \in G_1^i(X, f(X))$. Now let g be the right inverse of f. If $\alpha \in J(f)$, there is an associated map $H: X \times I \to X$ such that $H(x, 0) = H(x, 1) = f(x)$ and $[H(x, \ )] = \alpha$. Define $K: X \times I \to X$ be the map $K = H(g \times 1)$. Then $\alpha \in J(X)$. This completes the proof.

COROLLARY 10. *Let* $f: X \to X$ *be a homeomorphism from a connected compact polyhedron with non zero Euler characteristic* $\chi(X)$. *Then* $J(f) = 0$.

Proof. By the proposition 4.12[4], we have $J(X) = 0$. So that $J(f) = J(X) = 0$.

EXAMPLE 1. Let $f: S^1 \vee S^2 \to S^1 \vee S^2$ be a homeomorphism. Then $J(f) = 0$ because $\chi(S^1 \vee S^2) = 1$.

EXAMPLE 2. Let $X = S^1 \vee S^2$ and x_0 be the identifying base point. Let f be a selfmap of X given by $f(x) = x_0$ $(x \in S^2)$ and $f|_{S^1}: S^1 \to S^1$ is a homeomorphism. Then $G(X, f(X)) = G(X, S^1) = \pi_1(X)$. So that $J(f) = \pi_1(X)$. But $J(X) = 0$.

4. Nielsen number and the generalized evaluation subgroup

THEOREM 11. *Let* X *be a compact connected ANR,* $f: X \to X$ *be a map. Suppose there are integers* n, m *such that* $f_\pi^n(\pi_1(X)) \subset G_1^i(X, f^m(X))$ $(m \leq n)$. *Then all the fixed point classes of* f *have the same index. Hence*

$L(f) = 0$ *implies* $N(f) = 0$, *while*

$L(f) \neq 0$ *implies* $N(f) = \# coker(H_1(x) \xrightarrow{1-f_*} H_1(X))$.

Proof. It is sufficient to show that $f_\pi^n(\pi_1(X)) \subset J(f^n)$ by virtue of the results in [4] and [5]. Already we have easily proved that $G_1^i(X, f^m(X)) \subset G_1^i(X, f^n(X)) \subset J(f^n)$. This completes the proof.

EXAMPLE 3. Let $X = S^2 \vee \left(\bigvee_{i=1}^{m} S^1{}_i\right)$ and x_0 be the classifying base point. Suppose a map $f: (X, x_0) \to (X, x_0)$ satisfies

$$f(\alpha_1) = \alpha_1$$
$$f(\alpha_i) = \alpha_{i-1} \quad i > 1$$
$$f(x) = x_0 \quad x \in S^2$$

where α_i is a generator of $\pi_1(S^1{}_i, x_0) \subset \pi_1(X, x_0)$.
By the Theorem 2, we can have

$$G_1^i(X, f^m(X)) = Z(i_\pi \pi_1(f^m(X)),\ \pi_1(X)) = [\langle \alpha_1 \rangle]$$

So that $f_\pi^m(\pi_1(X)) = [\langle \alpha_1 \rangle] = G_1^i(X, f^m(X))$.

The above simple example show that the calculating $G_1^i(X, f^m(X))$ is sometimes better and easier than the calculating $J(f^n)$.

THEOREM 12. *Let X be a compact connected ANR, $f: X \to X$ be a map. Suppose there is an integer n such that $f_\pi^n(\pi_1(X))$ is abelian and $J(f^n) = \pi_1(X)$. Then any two fixed point classes of f have same index. Hence*

$$L(f) = 0 \textit{ implies } N(f) = 0, \textit{ while,}$$
$$L(f) \neq 0 \textit{ implies } N(f) = \# \textit{coker}\,(H_1(X) \xrightarrow{1-f^*} H_1(X))$$

Proof. The case $n = 1$, namely $f_\pi(\pi_1(X))$ is abelian and $J(f) = \pi_1(X)$. Combining two theorems-Theorem 4[1] and Theorem II 2.5 [4], we can easily have the required results.

In case $n > 1$. Although the technique of the proof in this case is quite similar to the proof of the main theorem in [5], we shall give the details of the proof. Now pick a prime q such that

(a) $f_\pi^q(\pi_1(X))$ is abelian and $J(f^q) = \pi_1(X)$,
(b) q is coprime to the order of the torsion subgroup of coker $(1 - f_*)$ and
(c) q is larger than the absolute value of the difference of indices for any two fixed point classes of f.

Such a q exists since if n satisfies condition in the hypothesis that so does any $n' > n$.

Let $F_i (i=1, 2)$ be two fixed point classes of f. The respective fixed point classes of f^q containing them are $F_i^{(q)}$. Then we have that different fixed point classes of f are contained in different fixed point classes of f^q, that is, $F_1^{(q)}=F_2^{(q)}$ implies $F_1=F_2$ by the assumption (a), (b) and the results in [4]. (cf. [5]).

Applying the results of which $n=1$ in view of the condition (a) for the map f^q, we get $\text{index}(f^q, F_1^{(q)})=\text{index}(f^q, F_2^{(q)})$.
By the mod p index theorem, ($fF_i^{(q)}=F_i^{(q)}$ since $F_i \subset F_i^{(q)}$), we have $\text{index}(f^q, F_i^{(q)})=\text{index}(f, F_i) \bmod q$ $(i=1, 2)$. Hence $\text{index}(f, F_1)=\text{index}(f, F_2) \bmod q$. But by the condition (c), we have $\text{index}(f, F_1) =\text{index}(f, F_2)$.

Corollary 13. *In the above theorem* 12, *we can replace* $G_1^i(X, f^n(X)) =\pi_1(X)$ *instead of* $J(f^n)=\pi_1(X)$.

Proof. Since $G_1^i(X, f^n(X)) \subset J(f^n) \subset \pi_1(X)$, $G_1^i(X, f^n(X))=\pi_1(X)$ implies $J(f^n)=\pi_1(X)$.

Example 4. Let X be $S^1 \vee S^2 \vee \cdots \vee S^n$ with identifying base point x_0. Let $f: (X, x_0) \to (X, x_0)$ be a map which satisfies that $[f|S^i]$ is a generator of $\pi_i(S^{i-1})$ if $i>1$ and $[f|S^1]$ is a nontrvial element of $\pi_1(S^1)$. Then $f^n(X)=S^1$ is a connected aspherical polyhedron. By the Theorem 2, we have

$$G_1^i(X, f^n(X))=Z(i_\pi \pi_1(f^n(X)), \ \pi_1(X))=\pi_1(X).$$

Thus any two fixed point class of f have the same index.

Corollary 14. *Let X be a compact connected polyhedron with the abelian fundamental group and without grobal separating point. Suppose X has a subspace A such that $G_1^i(X, A)=\pi_1(X)$. Then for any map $f: X \to X$ such that $f^n(X) \subset A$, $L(f)=0$ iff f is homotopic to a fixed point free map.*

Proof. Since $G_1^i(X, A) \subset G_1^i(X, f^n(X))$, it is clear by corollary 13 and the theorem II 6.1 [4].

Theorem 15. *Let $f: X \to X$ be a map on a connected compact aspherical polyhedron such that $f_\pi^n(\pi_1(X)) \subset Z(\pi_1(X))$. Then all the fixed point classes of f have the same index. So that*

$L(f)=0$ implies $N(f)=0$, while

$$L(f)\neq 0 \text{ implies } N(f)=\#\,coker(1-f_*).$$

Proof. It is well know that if $f: X\to X$ is a map from a connected aspherical polyhedron, then

$$Z(f_\pi(\pi_1(X)),\ \pi_1(X))=J(f).$$

So that, applying this and hypothesis to the map f^n, we can have

$$J(f^n)=\pi_1(X).$$

Hence we have the required results by virtue of our theorem 12.

COROLLARY 16. [5] *Let $f: X\to X$ be a map on a compact connected aspherical polyhedron such that $f_\pi^n(\pi_1(X))$ is abelian. Then all the fixed point classes of f have the same index.*

EXAMPLE 5. Let X be a klein bottle. The fundamental group $\pi_1(X)$ is isomorphic to a group with generators α and β satisfying the simple relation $\alpha\beta\alpha=\beta$. Furthermore X is a connected compact aspherical polyhedron. Let $f: X\to X$ be a map which induces the homomorphism f_π such that $f_\pi(\alpha)=1$, $f_\pi^n(\beta)=\beta^{2m}$ for some integers n, m. Since $\beta^2\in Z(\pi_1(X))$, $f_\pi^n(\pi_1(X))\subset Z(\pi_1(X))$. Thus we have $J(f^n)=\pi_1(X)$ so that all the fixed point classes of f have the same index.

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