J. Korean Math. Soc. **27**(1990), No. 2, pp. 193-202

HOMOGENEOUS POLYNOMIALS SATISFYING CAUCHY INTEGRAL EQUALITIES

JUN SOO CHOA AND HONG OH KIM

1. Introduction

Let $\mathcal{P}_n$ be the class of all holomorphic homogeneous polynomials π on C^n normalized so that

(1.1) $$\max \{|\pi(z)| : |z_1|^2+\cdots+|z_n|^2=1\}=1.$$

For $\pi\in\mathcal{P}_n$, if the sequence $C[\pi^{m+1}\bar{\pi}]$ of Cauchy integrals satisfies

(1.2) $$C[\pi^{m+1}\bar{\pi}]=\gamma_m\pi^m,\quad m=0,1,2,\cdots$$

for a sequence of positive numbers γ_m, then π is said to satisfy the Cauchy integral equalities, CIE for short (See [1,2]). Ahern and Rudin [1] noticed that if $\pi\in\mathcal{P}_n$ is a monomial or the sum-of-squares $(=z_1^2+\cdots+z_n^2)$ then it satisfies CIE and utilized this fact in their new proof of the BMOA-pullback theorem for such π. Choe [2] made more extensive study on CIE and asked whether there is a concrete characterization of $\pi\in\mathcal{P}_n$ satisfying CIE.

We observe that if $n=2$, the sum-of-squares

$$z_1^2+z_2^2=2\left(\frac{1}{\sqrt{2}}z_1-\frac{i}{\sqrt{2}}z_2\right)\left(\frac{1}{\sqrt{2}}z_1+\frac{i}{\sqrt{2}}z_2\right)$$

is obtained from the monomial $2w_1w_2$ in $\mathcal{P}_2$ by the unitary change of variables:

$$\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}.$$

This observation leads us to conjecture that if $\pi\in\mathcal{P}_2$ satisfies CIE then it can be transformed to a monomial by a unitary change of variables. We show in this paper that the conjecture is true for $\pi\in\mathcal{P}_2$ of degree ≤ 4. More precisely we prove

Received October 13, 1989.
Revised Feburuary 2, 1990.

THEOREM 1.1. *If $\pi\in\mathcal{P}_2$ of degree ≤ 4 satisfies* CIE *then it can be transformed to a monomial by a untary change of variables.*

The proof is very technical. The CIE condition (1.2) gives an infinite number of nonlinear algebraic equations on the coefficients of the homogeneous polynomial π. If the degree of π gets higher, the equations become too complicated to handle.

The second author wishes to express his sincere gratitude to Professor Patrick Ahern for helpful conversations during his visit to Madison last summer.

Any unexplained notations are as in [3].

2. Known results on CIE

We summarize some known results on CIE for $\mathcal{P}_n$ which will be used in the proof of Theorem 1.1.

PROPOSITION 2.1. [1, Lemma 2.2] *If $\pi(z)=b_\alpha z^\alpha\in\mathcal{P}_n$ or $\pi(z)=z_1^2+\cdots+z_n^2$ then π satisfies* CIE.

The following two propositions show that CIE holds only for very special polynomials in $\mathcal{P}_n$.

PROPOSITION 2.2. [2, 1, Remark 2.4] *$\pi(z)=a_1z_1^2+\cdots+a_nz_n^2$ ($a_i\neq 0$ for every i) satisfies* CIE *if and only if $|a_1|=\cdots=|a_n|=1$.*

PROPOSITION 2.3. [2, Example 3.8] *If $d\geq 2$ and $\pi(z)=a_1z_1^d+\cdots+a_nz_n^d$ ($|a_i|=1$ for every i) satisfies* CIE *then $d=2$.*

The following proposition gives a way of getting new polynomials satisfying CIE from an old one.

PROPOSITION 2.4. [2, Lemma 3.6] *If $\pi\in\mathcal{P}_n$ satisfies* CIE *and U is a unitary transformation of C^n then $\pi\circ U$ also satisfies* CIE.

3. Monomials and their unitary transforms

3.1. The unitary group $\mathcal{U}(2)$

We observe that any unitary matrix $U\in\mathcal{U}(2)$ of C^2 is of the form

$$(3.1)\qquad U=\begin{pmatrix}\mu\sqrt{1-r^2} & \lambda r\\ \nu r & -\bar{\mu}\lambda\nu\sqrt{1-r^2}\end{pmatrix},\quad |\lambda|=|\mu|=|\nu|=1,\ 0\leq r\leq 1,$$

which can be factored as $U=U_{\mu,\nu}V_rU_{1,\omega}(\omega=-\bar{\mu}\lambda)$, where

$$U_{\mu,\nu}=\begin{pmatrix}\mu & 0\\ 0 & \nu\end{pmatrix} \text{ and } V_r=\begin{pmatrix}\sqrt{1-r^2} & -r\\ r & \sqrt{1-r^2}\end{pmatrix}.$$

3.2. Monomials in $\mathcal{P}_2$ and their transforms

We note that if $\pi(z)=b_{lm}z_1^l z_2^m\in\mathcal{P}_2$ then

$$|b_{lm}|=\sqrt{\frac{d^d}{l^l m^m}}\quad (d=l+m)$$

by the normalization condition (1.1). By the unitary transformation corresponding to a suitable $U_{\mu,\nu}$ the monomial $\pi(z)=b_{lm}z_1^l z_2^m$ can be transformed to

$$\pi_{l,m}(z)=\sqrt{\frac{d^d}{l^l m^m}}z_1^l z_2^m\quad (d=l+m). \tag{3.2}$$

The unitary matrix

$$V_{\sqrt{\frac{m}{d}}}U_{1,\omega}=\begin{pmatrix}\sqrt{\frac{l}{d}} & -\sqrt{\frac{m}{d}}\omega\\ \sqrt{\frac{m}{d}} & \sqrt{\frac{l}{d}}\omega\end{pmatrix}$$

transforms $\pi_{l,m}$ again to

$$\tilde{\pi}_{l,m}=\sqrt{\frac{d^d}{l^l m^m}}\left(\sqrt{\frac{l}{d}}z_1-\sqrt{\frac{m}{d}}\omega z_2\right)^l\left(\sqrt{\frac{m}{d}}z_1+\sqrt{\frac{l}{d}}\omega z_2\right)^m, \tag{3.3}$$

which has value 1 at $(1,0)$ if $l\geq 1$. We list this correspondence in the following table (3.4) for later references.

(3.4)

d	$\pi_{l,m}(l\geq m)$	$\tilde{\pi}_{l,m}$
1	z_1	z_1
2	z_1^2	z_1^2
	$2z_1z_2$	$z_1^2-(\omega z_2)^2$
3	z_1^3	z_1^3
	$\frac{3\sqrt{3}}{2}z_1^2z_2$	$z_1^3-\frac{3}{2}z_1(\omega z_2)^2+\frac{1}{\sqrt{2}}(\omega z_2)^3$
4	z_1^4	z_1^4
	$\frac{16}{3\sqrt{3}}z_1^3z_2$	$z_1^4-2z_1^2(\omega z_2)^2+\frac{8}{3\sqrt{3}}z_1(\omega z_2)^3-\frac{1}{3}(\omega z_2)^4$
	$4z_1^2z_2^2$	$z_1^4-2z_1^2(\omega z_2)^2+(\omega z_2)^4$

5	z_1^5	z_1^5
	$\frac{25\sqrt{5}}{16}z_1^4z_2$	$z_1^5-\frac{5}{2}z_1^3(\omega z_2)^2+\frac{5}{2}z_1^2(\omega z_2)^3-\frac{15}{16}z_1(\omega z_2)^4+\frac{1}{8}(\omega z_2)^5$
	$\frac{25\sqrt{5}}{6\sqrt{3}}z_1^3z_2^2$	$z_1^5-\frac{5}{2}z_1^3(\omega z_2)^2+\frac{5\sqrt{6}}{18}z_1^2(\omega z_2)^3+\frac{5}{3}z_1(\omega z_2)^4-\frac{\sqrt{6}}{3}(\omega z_2)^5$

4. Proof of Theorem 1.1

PROPOSITOIN 4.1. *Any homogeneous polynomial $\pi\in\mathcal{P}_2$ of degree $d\geq 1$ can be transformed to π' of the form*

$$\pi'(z_1, z_2)=z_1^d+a_2z_1^{d-2}z_2^2+\cdots+a_dz_2^d$$

by a suitable unitary change of variables. (Note that the term $z_1^{d-1}z_2$ is missing in π')

Proof. Suppose $|\pi|$ attains its maximum at (ζ_1, ζ_2) on the sphere S_2 of C^2. Choose a unitary transform U which maps $(1,0)$ to (ζ_1, ζ_2) and set

$$\pi'(z_1, z_2)=\pi\circ U(z_1, z_2)=z_1^d+a_1z_1^{d-1}z_2+\cdots+a_dz_2^d.$$

Since $|\pi'|$ attains its maximum 1 at $(1,0)$ and the vector field $\frac{\partial}{\partial z_2}$ is tangential to S_2 at $(1,0)$, we have

$$0=\frac{\partial|\pi'|^2}{\partial z_2}(1,0)=\bar{\pi}'(1,0)a_1=a_1.$$

This completes the proof.

We now proceed on the proof of Theorem 1.1.

4.2. The case $d=1$ or $d=2$. By Proposition 4.1, any $\pi\in\mathcal{P}_2$, of degree 1 can be transformed, by a unitary transformation, to $\pi'(z)=z_1$, a monomial. By Proposition 4.1 again any $\pi\in\mathcal{P}_2$ of degree 2 can be transformed to $\pi'(z)=z_1^2+a_2z_2^2$ by a unitary transformation. If π satisfies CIE, then either $a_2=0$ or $|a_2|=1$ by Proposition 2.2. In either case, π' reduces to a monomial by a unitary transformation as we see in the table (3.4).

4.3. The case $d=3$. Suppose $\pi\in\mathcal{P}_2$ of degree 3 satisfies CIE. By Proposition 4.1, we may assume π is of the form

(4.1) $$\pi(z)=z_1^3+a_2z_1z_2^2+a_3z_2^3.$$

By another transformation corresponding to a suitable unitary matrix of the form $U_{1,\omega}$, we may assume $a_3\geq 0$. We compute $C[\pi^2\bar{\pi}]$ as follows.

We note that

$$(4.2)\qquad \pi(\zeta_1, \zeta_2)^2=\zeta_1^6+2a_2\zeta_1^4\zeta_2^2+2a_3\zeta_1^3\zeta_2^3 +a_2^2\zeta_1^2\zeta_2^4+2a_2a_3\zeta_1\zeta_2^5+a_3^2\zeta_2^6,$$

and

$$(4.3)\qquad \bar{\pi}(\zeta_1, \zeta_2)\langle z, \zeta\rangle^3=z_1^3(\bar{\zeta}_1^6+\bar{a}_2\bar{\zeta}_1^4\bar{\zeta}_2^2+\bar{a}_3\bar{\zeta}_1^3\bar{\zeta}_2^3) +3z_1^2z_2(\bar{\zeta}_1^5\bar{\zeta}_2+\bar{a}_2\bar{\zeta}_1^3\bar{\zeta}_2^3+\bar{a}_3\bar{\zeta}_1^2\bar{\zeta}_2^4) +3z_1z_2^2(\bar{\zeta}_1^4\bar{\zeta}_2^2+\bar{a}_2\bar{\zeta}_1^2\bar{\zeta}_2^4+\bar{a}_3\bar{\zeta}_1\bar{\zeta}_2^5) +z_2^3(\bar{\zeta}_1^3\bar{\zeta}_2^3+\bar{a}_2\bar{\zeta}_1\bar{\zeta}_2^5+\bar{a}_3\bar{\zeta}_2^6).$$

We use the orthogonality relations for monomials [3, Propositions 1.4.8 and 1.4.9] in the following computation of the Cauchy integral. From (4.2) and (4.3), we have

$$(4.4)\qquad \begin{aligned} C[\pi^2\bar{\pi}](z) &= \int_{S_2}\frac{\pi^2(\zeta)\bar{\pi}(\zeta)}{(1-\langle z, \zeta\rangle)^2}d\sigma(\zeta) \\ &= \sum_{j=0}^{\infty}\binom{-2}{j}(-1)^j\int_{S_2}\pi^2(\zeta)\bar{\pi}(\zeta)\langle z, \zeta\rangle^j d\sigma(\zeta) \\ &= \binom{-2}{3}(-1)^3\int_{S_2}\pi^2(\zeta)\bar{\pi}(\zeta)\langle z, \zeta\rangle^3 d\sigma(\zeta) \\ &= 4\Big\{\Big(\frac{6!}{7!}+2|a_2|^2\frac{4!2!}{7!}+2|a_3|^2\frac{3!3!}{7!}\Big)z_1^3 \\ &\quad +3\Big(2\bar{a}_2a_3\frac{3!3!}{7!}+a_2^2\bar{a}_3\frac{2!4!}{7!}\Big)z_1^2z_2 \\ &\quad +3\Big(2a_2\frac{4!2!}{7!}+a_2|a_2|^2\frac{2!4!}{7!}+2a_2|a_3|^2\frac{5!}{7!}\Big)z_1z_2^2 \\ &\quad +\Big(2a_3\frac{3!3!}{7!}+2|a_2|^2a_3\frac{5!}{7!}+a_3|a_3|^2\frac{6!}{7!}\Big)z_2^3\Big\} \\ &= \frac{4\cdot 4!}{7!}\{(30+4|a_2|^2+3|a_3|^2)z_1^3 \\ &\quad +3(3\bar{a}_2a_3+2a_2^2\bar{a}_3)z_1^2z_2 \\ &\quad +3(4a_2+2a_2|a_2|^2+10a_2|a_3|^2)z_1z_2^2 \\ &\quad +(3a_3+10|a_2|^2a_3+30|a_3|^2a_3)z_2^3\}. \end{aligned}$$

Comparing the coefficients in the CIE condition $C[\pi^2\bar{\pi}]=\gamma_1\pi$, we have the following equations from (4.1) and (4.4). Recall that $a_3\geq 0$ is assumed.

(a. 0) $30+4|a_2|^2+3a_3^2=\tilde{\gamma}_1$,

(a. 1) $3\bar{a}_2a_3+2a_2^2a_3=0$,

(a. 2) $3a_2(4+2|a_2|^2+10a_3^2)=a_2\tilde{\gamma}_1$,

(a. 3) $a_3(3+10|a_2|^2+30a_3^2)=a_3\tilde{\gamma}_1$.

($\tilde{\gamma}_1>0$ is another constant)

Case 1. $a_3=0$: Suppose $a_2\neq 0$. We solve (a. 0) and (a. 2) for $|a_2|$ and get $|a_2|=3$. If we set $a_2=3\omega$ with $|\omega|=1$, then

$$\pi\left(\frac{1}{\sqrt{2}}, \frac{\bar{\omega}}{\sqrt{2}}\right)=\left(\frac{1}{\sqrt{2}}\right)^3+3\omega\left(\frac{1}{\sqrt{2}}\right)^2\frac{\bar{\omega}}{\sqrt{2}}=\frac{4}{2\sqrt{2}}>1,$$

which is impossible for $\pi\in\mathcal{P}_2$. Therefore $a_2=0$ and $\pi(z)=z_1^3$, a monomial.

Case 2. $a_3\neq 0$: Suppose $a_2=0$. We solve (a. 0) and (a. 3) for a_3 and get $a_3=1$. But $\pi(z)=z_1^3+z_2^3$ cannot satisfy CIE by Proposition 2. 3. We should have $a_2\neq 0$. From (a. 1), we have $a_2=\frac{3}{2}\omega$ with $\omega^3=-1$.

Now, we solve (a. 0) and (a. 2) to get $a_3=\frac{1}{\sqrt{2}}$.

Therefore, we have

$$\begin{aligned}\pi(z)&=z_1^3+\frac{3}{2}\omega z_1z_2^2+\frac{1}{\sqrt{2}}z_2^3\\ &=z_1^3-\frac{3}{2}z_1(\omega^2z_2)^2+\frac{1}{\sqrt{2}}(\omega^2z_2)^3,\end{aligned}$$

which can be transformed to the monomial $\frac{3\sqrt{3}}{2}z_1^2z_2$ by a suitable unitary transformation as we see in the table (3. 4).

4. 4. The case $d=4$. Suppose $\pi\in\mathcal{P}_2$ of degree 4 satisfies CIE. By Proposition 4. 1 we may assume π is of the form

$$\pi(z)=z_1^4+b_2z_1^2z_2^2+b_3z_1z_2^3+b_4z_2^4.$$

From the CIE condition $C[\pi^2\bar{\pi}]=\gamma_1\pi$, we have the following equations on the coefficients as before.

(b. 0) $140+10|b_2|^2+5|b_3|^2+2(b_2^2+2b_4)\bar{b}_4=\tilde{\gamma}_1$,

(b. 1) $5b_3\bar{b}_2+2(b_2^2+2b_4)\bar{b}_3+5b_2b_3\bar{b}_4=0$,

(b. 2) $60b_2+12(b_2^2+2b_4)\bar{b}_2+30b_2|b_3|^2+30(b_3^2+2b_2b_4)\bar{b}_4=b_2\tilde{\gamma}_1$,

(b. 3) $20b_3+20|b_2|^2b_3+20(b_3^2+2b_2b_4)\bar{b}_3+140b_3|b_4|^2=b_3\tilde{\gamma}_1$,

(b. 4) $2(b_2^2+2b_4)+5(b_3^2+2b_2b_4)\bar{b}_2+35|b_3|^2b_4+140|b_4|^2b_4=b_4\tilde{\gamma}_1$.

($\tilde{\gamma}_1>0$ a constant)

Case 1. $b_3=0$, $b_4=0$: From (b. 4), $b_2=0$; so $\pi(z)=z_1^4$, a monomial.

Case 2. $b_3=0$, $b_4\neq 0$: Suppose $b_2=0$. From (b. 0) and (b. 4), we have $|b_4|=1$, which is impossible by Proposition 2. 3. We should have $b_2\neq 0$. We may assume that $b_2=$real by a suitable unitary transformation. We solve (b. 0), (b. 2) and (b. 4) for b_4 and get $b_4=1$; so $\pi(z)=z_1^4+b_2z_1^2z_2^2+z_2^4$. Now we have to consider CIE condition for $m=2$. Comparing the coefficients of z_1^8 and $z_1^6z_2^2$ in $C[\pi^3\bar{\pi}]=\gamma_2\pi^2$, we have

$$332+17b_2^2=\tilde{\gamma}_2,$$
$$268+33b_2^2=\tilde{\gamma}_2, \quad \tilde{\gamma}_2>0,$$

from which $b_2=\pm 2$. Therefore we have $\pi(z)=z_1^4\pm 2z_1^2z_2^2+z_2^4$, which can be transformed to the monomial $4z_1^2z_2^2$ by a unitary transformation as seen in the table (3.4).

Case 3. $b_3\neq 0$, $b_2=0$: From (b.1), $b_4=0$. We show that this case cannot happen. We note that

$$\pi^{m+1}(\zeta)=(\zeta_1^4+b_3\zeta_1\zeta_2^3)^{m+1}$$
$$=b_3^{m+1}\zeta_1^{m+1}\zeta_2^{3m+3}+(m+1)b_3^m\zeta_1^{m+4}\zeta_2^{3m}+\cdots$$

and

$$\pi(\zeta)\zeta_1^m\zeta_2^{3m}=\zeta_1^{m+4}\zeta_2^{3m}+b_3\zeta_1^{m+1}\zeta_2^{3m+3}.$$

From the orthogonality relations for monomials [3, Propositions 1.4.8 and 1.4.9], we have

$$\int_{S_2}\pi^{m+1}(\zeta)\bar{\pi}(\zeta)\bar{\zeta}_1^m\bar{\zeta}_2^{3m}d\sigma(\zeta)$$
$$=b_3^m|b_3|^2\frac{(m+1)!\,(3m+3)!}{(4m+5)!}+(m+1)b_3^m\frac{(m+4)!\,(3m)!}{(4m+5)!},$$

and

$$\int_{S_2}\pi^m(\zeta)\bar{\zeta}_1^m\bar{\zeta}_2^{3m}d\sigma(\zeta)=b_3^m\frac{m!\,(3m)!}{(4m+1)!}.$$

Since $b_3\neq 0$, the CIE condition (1.2) implies that

$$\gamma_m=\frac{(m+1)(3m+3)(3m+2)(3m+1)}{(4m+5)(4m+4)(4m+3)(4m+2)}|b_3|^2$$
$$+\frac{(m+1)(m+4)(m+3)(m+2)(m+1)}{(4m+5)(4m+4)(4m+3)(4m+2)}.$$

This is contradictory to the fact that $\gamma_m\to 1$ as $m\to\infty$. See [1, page 135].

Case 4. $b_3\neq 0$, $b_2\neq 0$: We may assume $b_3>0$ by a unitary transformation. (b.1) and (b.3) then reduce respectively to

(b.1)′ $5\bar{b}_2+2(b_2^2+2b_4)+5b_2\bar{b}_4=0,$

(b.3)′ $20+20|b_2|^2+20b_3^2+40b_2b_4+140|b_4|^2=\tilde{\gamma}_1.$

From (b.0) and (b.3)′, we should have

(4.5) $$b_2^2\bar{b}_4=\text{real and } b_2b_4=\text{real}.$$

If we set $b_2=\rho\omega$ with $\rho>0$ and $|\omega|=1$ and set $t=b_2^2\bar{b}_4$ (=real), then

(4.6) $$b_4=\frac{t}{\rho^2}\omega^2,$$

and

(4.7) $$b_2b_4=\frac{t}{\rho}\omega^3 \text{ real};$$

so $\omega^3=\pm 1$. The equations (b. 0), (b. 1)′, (b. 2), (b. 3)′ and (b. 4) then can be written respectively as follows.

(b. 0)″ $140+10\rho^2+5b_3{}^2+2t+4\frac{t^2}{\rho^4}=\tilde{\gamma}_1$,

(b. 1)″ $5\rho+2\left(\rho^2+\frac{2t}{\rho^2}\right)\omega^3+5\frac{t}{\rho}=0$,

(b. 2)″ $60+12\rho^2+24\frac{t}{\rho^2}+30b_3{}^2+30b_3{}^2\frac{t}{\rho^3}\omega^3+60\frac{t^2}{\rho^4}=\tilde{\gamma}_1$,

(b. 3)″ $20+20\rho^2+20b_3{}^2+40\frac{t}{\rho}\omega^3+140\frac{t^2}{\rho^4}=\tilde{\gamma}_1$,

(b. 4)″ $\frac{2\rho^4}{t}+4+5b_3{}^2\frac{\rho^3}{t}\omega^3+10\rho^2+35b_3{}^2+140\frac{t^2}{\rho^4}=\tilde{\gamma}_1$.

We consider the cases $\omega^3=1$ and $\omega^3=-1$ separately.
Subcase 1. $\omega^3=1$: From (b. 1)″, we have

(4.8) $$t=-\frac{\rho^3(2\rho+5)}{5\rho+4}.$$

If we eliminate $\tilde{\gamma}_1$ from (b. 0)″ and (b. 3)″ and substitute (4.8) in place of t, we get

(4.9) $15(5\rho+4)^2b_3{}^2=-20\rho^5-460\rho^4-1840\rho^3+240\rho^2+4800\rho+1920$.

From (b. 3)″, (b. 4)″ and (4.8), we have

(4.10) $5(\rho+11)(5\rho+4)b_3{}^2=-60\rho^4-420\rho^3-560\rho^2+560\rho+320$.

From (b. 0)″, (b. 2)″ and (4.8), we again have

(4.11)
$5(5\rho+4)(13\rho-10)b_3{}^2=-20\rho^5-340\rho^4-1000\rho^3+1360\rho^2+3680\rho+1280$.

If we eliminate $b_3{}^2$ either from (4.9) and (4.10) or from (4.9) and (4.11), we have

(4.12) $\rho^6-11\rho^5-6\rho^4+328\rho^3-288\rho^2-2160\rho-864=0$.

The equation (4.12) can be factored as

(4.13) $(\rho+2)(\rho-6)^3(\rho^2+5\rho+2)=0$.

Since $\rho>0$, $\rho=6$. (4.10) then reduces to

$$2890b_3{}^2=-184960,$$

which is impossible since $b_3{}^2>0$. Therefore the case $\omega^3=1$ cannot happen.

Subcase 2. $\omega^3=-1$: From (b. 1)″, we have

$$(5\rho-4)t=\rho^3(2\rho-5),$$

which implies $5\rho-4\neq 0$ and

$$(4.14)\qquad t=\frac{\rho^3(2\rho-5)}{5\rho-4}.$$

If we eliminate $\tilde{\gamma}_1$ and t from (b. 0)″, (b. 3)″ and (4. 14), we have

$$(4.15)\qquad 15(5\rho-4)^2b_3^2=20\rho^5-460\rho^4+1840\rho^3+240\rho^2-4800\rho+1920.$$

Eliminating $\tilde{\gamma}_1$ and t from (b. 3)″, (b. 4)″ and (4. 14), we get

$$(4.16)\qquad 5(\rho-11)(5\rho-4)b_3^2=-60\rho^4+420\rho^3-560\rho^2-560\rho+320.$$

If we eliminate b_3^2 from (4. 15) and (4. 16) we have

$$(4.17)\qquad \rho^6+11\rho^5-6\rho^4-328\rho^3-288\rho^2+2160\rho-864=0.$$

The equation (4. 17) can be factored as

$$(4.18)\qquad (\rho-2)(\rho+6)^3(\rho^2-5\rho+2)=0.$$

Therefore we have either $\rho=2$ or $\rho^2=5\rho-2$. If $\rho^2=5\rho-2$, then the right hand side of (4. 15) becomes zero; so $b_3^2=0$, a contradiction. Therefore $\rho=2$. We then have

$$t=-\frac{4}{3},$$

$$b_3^2=\frac{64}{27}, \text{ or } b_3=\frac{8}{3\sqrt{3}}$$

$$b_4=-\frac{1}{3}\omega^2.$$

We have then

$$\begin{aligned}\pi(z)&=z_1^4+2\omega z_1^2z_2^2+\frac{8}{3\sqrt{3}}z_1z_2^3-\frac{1}{3}\omega^2z_1^4\\&=z_1^4-2z_1^2(\omega^2z_2)^2+\frac{8}{3\sqrt{3}}z_1(\omega^2z_2)^3-\frac{1}{3}(\omega^2z_1)^4,\end{aligned}$$

which can be transformed to the monomial $\frac{16}{3\sqrt{3}}z_1^3z_2$ by a suitable unitary transformation as in the table (3. 4). This completes the proof of the Theorem 1. 1.

Acknowledgement. The authors wish to express their sincere thanks to the referee for pointing out numerical errors.

References

1. P. Ahern and W. Rudin, *Bloch functions, BMO and boundary zeros,* Indiana Math. J. **36** (1987), 131-148.
2. B.R. Choe, *Cauchy integral equalities and applications,* Trans. Amer. Math. Soc. **315** (1989), 337-352.
3. W. Rudin, Function theory in the unit ball of C^n, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

Sung Kyun Kwan University
Seoul 110-745, Korea
and
KAIST
Seoul 136-791, Korea