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FIXED POINTS OF CONDENSING INWARD MULTIFUNCTIONS

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1. Introduction

The Brouwer or Kakutani type fixed point theorems have been extended for various types of condensing multifunctions. In [7], W. V. Petryshyn and P. M. Fitzpatrick noted how a fixed point theorem for certain condensing multifunctions could be obtained from the corresponding results for multifunctions with compact domains.

Recently, the present author [6] extended the Brouwer-Kakutani type theorems to a class of multifunctions more general than upper hemicontinuous ones defined on convex subsets of a real topological vector space having sufficiently many linear functionals. Moreover, those multifunctions have more general boundary conditions than the weakly inward (outward) ones.

The purpose in the present paper is to obtain fixed point theorems for certain condensing multifunctions with the inwardness condition. We use the recent results in [6]. Consequently, key results of Petryshyn and Fitzpatrick [7] are extended for a larger class of multifunctions properly containing upper hemicontinuous ones and, in certain cases, for more general topological vector spaces than locally convex ones.

2. Preliminaries

Let D and E be two sets. A *multifunction* $F : D \to 2^E$ is a function from D into the power set 2^E of E, that is, $Fx \subset E$ for each $x \in D$. Let $F(D) := \bigcup \{Fx : x \in D\}$.

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Let E be a real Hausdorff topological vector space (t. v. s.) and E^* the topological dual of E. A t. v. s. E is said to ***have sufficiently many linear functionals*** if E^* separates points of E (that is, for every $x \in E$ with $x \neq 0$, there exists an $f \in E^*$ such that $fx \neq 0$). By the Hahn-Banach theorem, every locally convex t. v. s. (l. c. s.) has sufficiently many linear functionals, but not conversely.

Let D be a topological space, E a t. v. s., and $F : D \to 2^E$. Then

(i) F is *upper semicontinuous* (*u. s. c.*) if for each $y_0 \in D$ and for each open set U in E containing Fy_0, there exists an open neighborhood N of y_0 in D such that $F(N) \subset U$;

(ii) F is *upper demicontinuous* (*u. d. c.*) if for each $y_0 \in D$ and for each open half-space H in E containing Fy_0, there exists an open neighborhood N of y_0 in D such that $F(N) \subset H$; and

(iii) F is *upper hemicontinuous* (*u. h. c.*) if for $f \in E^*$ and for any real a, the set

$$\{y \in D : \sup f(Fu) < a\}$$

is open in D.

Note that u. s. c. $\Longrightarrow$ u. d. c. $\Longrightarrow$ u. h. c. and that if the multifunction is compact valued, then u. d. c. $\Longleftrightarrow$ u. h. c. If $F, G : D \to 2^E$ are u. h. c., then so is $F+G$.

If F is u. h. c., then the set $\{y \in D : \sup f(Fy) < fy\}$ is open for any $f \in E^*$, but not conversely [6].

Let $cc(E)$ denote the set of nonempty closed convex subsets of E and $kc(E)$ the set of nonempty compact convex subsets of E. Bd, $-$, and co denote the boundary, closure, and convex hull, resp., with respect to E.

Let $D \subset E$ be convex and $x \in E$. The *inward* and *outward sets of* D at x, $I_D(x)$ and $O_D(x)$, are defined as follows:

$$I_D(x) = x + \bigcup_{r>0} r(D-x), \quad O_D(x) = x + \bigcup_{r<0} r(D-x).$$

A multifunction $F : D \to 2^E$ is said to be *inward* (*outward*, resp.) if $Fx \cap I_D(x) \neq \emptyset$ [$Fx \cap O_D(x) \neq \emptyset$, resp.] for each $x \in \mathrm{Bd}\, D \setminus Fx$, and *weakly inward* (*outward*, resp.) if $Fx \cap \bar{I}_D(x) \neq \emptyset$ [$Fx \cap \bar{O}_D(x) \neq \emptyset$, resp.] for each $x \in \mathrm{Bd}\, D \setminus Fx$.

For $f \in E^*$ and $U, V \subset E$, let

$$d_f(U, V) := \inf\{|f(u-v)| : u \in U, \ v \in V\}.$$

If F is weakly inward (outward, resp.), then

$$d_f(Fx, \bar{I}_D(x)) = 0 \quad (d_f(Fx, \bar{O}_D(x)) = 0, \text{ resp.})$$

holds for any $f \in E^*$ and $x \in \mathrm{Bd}\, D \backslash Fx$, but not conversely. See Jiang [2].

A multifunction $F : D \to 2^E$ is said to be *generalized condensing* if whenever $Q \subset D$, $F(Q) \subset Q$, and $Q \backslash \overline{\mathrm{co}} F(Q)$ is relatively compact, then Q is relatively compact.

If C is a lattice with a minimal element, which we denote by 0, a function $\Phi : 2^E \to C$ is called a *measure of noncompactness* if

(1) $\Phi(\overline{\mathrm{co}} Q) = \Phi(Q)$ for each $Q \in 2^E$,

(2) $\Phi(Q) = 0$ iff Q is precompact, and

(3) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$ for each $A, B \in 2^E$.

For example, the set-measure of noncompactness γ and the ball-measure of noncompactness χ satisfy (1)-(3). See [7].

A multifunction $F : D \to 2^E$ is said to be *Φ-condensing* provided that if $Q \subset D$ with $\Phi(F(Q)) \geq \Phi(Q)$, then Q is relatively compact.

The following are known [7]:

(i) Every multifunction defined on a compact set is Φ-condensing.

(ii) If E is a l. c. s., every compact multifunction $F : D \to 2^E$ is Φ-condensing if either D is complete or E is quasi-complete.

(iii) Every compact or Φ-condensing multifunction is generalized condensing.

3. Basic results

The first two statements in this section are due to Petryshyn and Fitzpatrick [7]. The authors assumed local convexity of the underlying t. v. s. E, which is not necessary by closely examining their proofs.

Lemma 3. 1. *Let E be a t. v. s., D a closed convex subset of E, and $F : D \to 2^E$.*

(1) [7, *Lemma* 1] *If $Q \subset D$, then there exists a closed convex set $K = K(F, D, Q)$ with $Q \subset K$ and $\overline{\mathrm{co}}\, \{F(D \cap K) \cup Q\} = K$.*

(2) *If F is inward, then so is $F|D \cap K$.*

Proof of (2). For any $x \in D \cap K$, there exist $z \in Fx$ and $y \in D$ such that $z = x + r(y-x) \in I_D(x)$ for some $r > 0$, since F is inward and $x \in D$. Note that $z \in Fx \subset F(D \cap K) \subset K$. If $1 > r > 0$, $z = (1-r)x + ry \in D$ since D is convex. Hence, $z \in D \cap K$ and $z \in Fx \cap (D \cap K) \subset Fx \cap I_{D \cap K}(x) \neq \emptyset$. If $r \geq 1$, then $y = r^{-1}z + (1-r^{-1})x \in K$ since $z, x \in K$ and K is convex. Hence, $y \in D \cap K$ and $z = x + r(y-x) \in Fx \cap I_{D \cap K}(x) \neq \emptyset$.

A multifunction $F : D \to 2^E$ is said to be *ultimately compact* if $F(D \cap K)$ is relatively compact where $K = K(F, D, A)$ and A is any precompact subset of D [7]. It is known that [7] every generalized condensing multifunction is ultimately compact.

LEMMA 3.2. [7, *Proposition* 4] *Let E be a t.v.s., D a closed convex subset of E, and $F : D \to 2^E$ a Φ-condensing inward multifunction. Then for each precompact $Q \subset D$, there exists a compact convex set $A \subset E$ such that $Q \subset A$ and F is inward on A.*

The following are given in our previous work [6].

THEOREM 3.3. *Let E be a t.v.s., K a nonempty compact convex subset of E, and $F : K \to 2^E$ a multifunction such that $\{x \in K : \sup f(Fx) < fx\}$ is open in K for any $f \in E^*$ and that either*

(A) E^* separates points of E and $F : K \to kc(E)$, or

(B) E is locally convex and $F : K \to cc(E)$.

(1) *If $d_f(Fx, \bar{I}_K(x)) = 0$ for any $f \in E^*$ and $x \in Bd\, K \backslash Fx$, then F has a fixed point.*

(2) *If $d_f(Fx, \bar{O}_K(x)) = 0$ for any $f \in E^*$ and $x \in Bd\, K \backslash Fx$, then F has a fixed point. Further, if F is u.h.c., then $K \subset F(K)$.*

Note that Theorem 3.3 generalizes a number of well-known Brouwer-Kakutani type fixed point theorems. See [5], [6].

THEOREM 3.4. *Let E be a t.v.s., K a nonempty compact convex subset of E, and $F, G : K \to 2^E$ multifunctions. Suppose that either*

(A) E^* separates points of E and $F, G : K \to kc(E)$, or

(B) E is locally convex and $F, G : K \to cc(E)$ such that Fx or Gx is compact for each $x \in K$.

Suppose that $S := I + F - G$, where $I : K \to E$ is the inclusion, satisfies the following : for each $f \in E^*$,

(a) $\{x \in K : \sup f(Sx) < fx\}$ is open in K, and

(b) $d_f(Sx, \bar{I}_K(x)) = 0$ for each $x \in Bd\, K \backslash Fx$.

Then F and G have a coincidence point.

Note that if F and G are u.h.c., then so is S in Theorem 3.4. Hence, Lee and Tan [4, Theorem 6] follows from Theorem 3.4 (B).

The following is a consequence of Theorem 3.4(B).

COROLLARY 3.5 *Let E be a l.c.s., K a nonempty closed bounded convex subset of E, and $F, G: K \to cc(E)$ multifunctions such that Fx or Gx is compact for each $x \in K$. If $S := I + F - G$ is a Φ-condensing multifunction satisfying (a), (b), in Theorem 3.4, and*

(c) K is contained in $co(R(K) \cup Q)$ for some relatively compact subset Q of E,

then F and G have a coincidence point.

Proof. As in [3, Lemma], (c) implies that K is compact.

Note that Kim and Jeoung [3, Theorems 2–4] are consequences of Corollary 3.5.

4. Main results

In this section, we improve results in [7].

THEOREM 4.1. *Let E be a t.v.s. and $D \in cc(E)$. Suppose that either*

(A) E^* separates points of E and $F, G: D \to kc(E)$, or

(B) E is locally convex, and $F, G: D \to cc(E)$ such that Fx or Gx is compact for each $x \in D$.

If $S := I + F - G$ is inward, the set $\{x \in D : \sup f(Sx) < fx\}$ is open in D for each $f \in E^*$, and $D \cap K$ is compact for $K = K(S, D, A)$ where A is a nonempty subset of D, then F and G have a coincidence point.

Proof. Note that $D \cap K \in kc(E)$. By Lemma 3.1, $S | D \cap K$ is inward. Hence, the conclusion follows from Theorem 3.4.

In [7, Theorem 1], the authors obtained the case (B) of Theorem 4.1 for u.d.c. multifunctions F, G.

THEOREM 4.2. *Let E, D, and F, G be as in Theorem 4.1. If $S := I + F - G$ is generalized condensing with $S(D) \subset D$ and $\{x \in D : \sup f(Sx) < fx\}$ is open in D for each $f \in E^*$, then F and G have a coincidence point.*

Proof. By the argument in the proof of [7, Proposition 1], there exists a nonempty compact convex set $K \subset D$ satisfying $\overline{\mathrm{co}}\, S(K) = K$. Then $S | K$ has a fixed point by Theorem 3.4 or 4.1.

In [7, Proposition 1], the authors obtained the case (B) of Theorem 4.2 for an u. d. c. multifunction S.

THEOREM 4.3. *Let E, D, and F, G be as in Theorem 4.1. If $S:=I+F-G$ is Φ-condensing and inward, and $\{x\in D : \sup f(Sx)<fx\}$ is open in D for each $f\in E^*$, then F and G have a coincidence point.*

Proof 1. As in the proof of [7, Proposition 2], the set $D\cap K$ in Lemma 3.1 can be seen compact, where $K=K(S, D, \{x_0\})$ for some $x_0\in D$. Therefore, by Theorem 4.1, we have the conclusion.

Proof 2. For any precompact set $Q\subset D$, say $Q=\{x_0\}$ for some $x_0\in D$, by Lemma 3.2, we have a compact convex set $A\subset D$ such that $S|A$ is inward. Now, by Theorem 3.4, we have the conclusion.

The case (B) of Theorem 4.3 for u. d. c. multifunctions F and G reduces to [7, Proposition 2].

By putting $G=I$ in Theorems 4.2 and 4.3, we obtain the following fixed point result:

THEOREM 4.4. *Let E be a t. v. s., $D\in cc(E)$, and $F : D\to 2^E$. Suppose that $\{x\in D : \sup f(Fx)<fx\}$ is open for each $f\in E^*$, and that either*

(A) E^* separates points of E and $F : D\to kc(E)$, or

(B) E is locally convex and $F : D\to cc(E)$.

If either F is Φ-condensing and inward or F is generalized condensing and $F(D)\subset D$, then F has a fixed point.

For an u. d. c. multifunction F, Theorem 4.4(B) is due to Petryshyn and Fitzpatrick [7, Corollary 2].

THEOREM 4.5. *Let E be a quasi-complete l. c. s. and $D\in cc(E)$. If $F : D\to cc(E)$ is ultimately compact and inward, and $\{x\in D : \sup f(Fx)<fx\}$ is open for each $f\in E^*$, then F has a fixed point.*

Proof. By Theorem 4.1(B) with $G=I$, it suffices to show that $D\cap K$ is compact for $K=K(F, D, \{x_0\})$ and $x_0\in D$. Note that $F(D\cap K)$ is relatively compact and $K=\overline{\mathrm{co}}\{F(D\cap K)\cup\{x_0\}\}$. Since in a quasi-complete l. c. s., the convex closure of a precompact set is compact, K is compact, and so is $D\cap K$.

For an u. d. c. multifunction F, Theorem 4. 5 reduces to [7, Proposition 3].

As noted in [7], Theorems 4. 4 and 4. 5 contain earlier generalizations of the Brouwer or Kakutani fixed point theorem due to Browder, Ky Fan, Glicksberg, Bohnenblust and Karlin, Halpern, Reich, Daneŝ, and Himmelberg et al. See [7] and [5].

It is open whether Theorems 4. 4 or 4. 5 hold for a weakly inward multifunction F. It is also open whether Theorem 4. 5 holds for a t. v. s. E on which E^* separates points.

Note that for a real Banach space E and an u. s. c. γ-condensing multifunction F, Theorem 4. 4 holds for the weakly inward case. This is due to K. Deimling [1].

For the definition of k-Φ-contraction, see [7].

Finally, we note that [7, Theorem 2] holds for more general setting as follows:

THEOREM 4. 6. *Let Φ be either γ or χ, E a l. c. s., and $D \in cc(E)$ such that either E is quasi-complete or D is complete. Suppose that $F: D \to cc(E)$ is a 1-Φ-contraction and inward, and $\{x \in D : \sup f(Fx) < fx\}$ is open for each $f \in E^*$, and assume that whenever $\{x_n\} \subset D$ with $y_n \in Fx_n$ for each n and $x_n - y_n \to 0$, then there exists an $x \in D$ with $x \in Fx$. Then F has a fixed point.*

Proof. Just follow the proof of [7, Theorem 2] using Theorems 4. 1 and 4. 4.

Note that [7, Theorem 2] is just Theorem 4. 6 for an u. d. c. F, and that [7, Corollary 3] also holds for u. h. c. multifunctions.

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