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TRANSVERSE FIELDS PRESERVING THE TRANSVERSE RICCI FIELD OF A FOLIATION

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1. Introduction

Let $(M, g, \mathcal{F})$ be a connected, orientable, closed, $p+q$ dimensional Riemannian manifold with a transversally orientable foliation $\mathcal{F}$ of codimension q and a bundle-like metric g with respect to $\mathcal{F}$. Let Ric_D be the transverse Ricci field of $\mathcal{F}$ with respect to the transverse Riemannian connection D, where D is a torsion-free and g_Q-metrical connection on the normal bundle Q of $\mathcal{F}$. We consider the transverse projective (or, conformal, Killing) fields of $\mathcal{F}$. It is trivial that a transverse Killing field s of $\mathcal{F}$ preserves the transverse Ricci field of $\mathcal{F}$, that is, $\Theta(s)\mathrm{Ric}_D=0$. Here we denote by $\Theta(s)$ the transverse Lie differentiation with respect to s. The purpose of this note is to prove the following theorems:

THEOREM A. *Let $(M, g, \mathcal{F})$ be as above. Suppose that $\mathcal{F}$ is harmonic and $q\geq 2$. If a transverse conformal field s of $\mathcal{F}$ satisfies $\Theta(s)Ric_D=0$, then s is a transverse Killing field of $\mathcal{F}$.*

THEOREM B. *Let $(M, g, \mathcal{F})$ be as above. Suppose that $\mathcal{F}$ is harmonic and $q\geq 2$. If a transverse projective field s of $\mathcal{F}$ satisfies $\Theta(s)Ric_D=0$, then s is a transverse Killing field of $\mathcal{F}$.*

If we consider the case of foliation by points, then the above theorems imply the well-known results ([1]): Let (M, g) be a connected, orientable and closed Riemannian manifold, and let Ric be the Ricci tensor field of type $(0, 2)$ on M. If a conformal (or, projective) vector field Y on M satisfies $\theta(Y)\mathrm{Ric}=0$, then Y is a Killing vector field on M, where $\theta(Y)$ denotes the Lie differentiation with respect to Y.

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We shall be in C^∞-category and deal only with connected, orientable, and closed (=compact and without boundary) manifolds. We assume that foliations are transversally orientable. We use the following convention on the range of indices: $1 \leq i, j, \cdots \leq p$, $p+1 \leq \alpha, \beta, \cdots \leq p+q$. The Einstein summation convention will be used.

2. Preliminaries

Let $(M, g, \mathcal{F})$ be a $p+q$ dimensional foliated Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g in the sense of Reinhart [8]. The foliation $\mathcal{F}$ defines an integrable subbundle E of the tangent bundle TM over M. The quotient bundle $Q=TM/E$ is called the normal bundle of $\mathcal{F}$. Let $\pi: TM \longrightarrow Q$ be the natural projection. The metric g defines a map $\sigma: Q \longrightarrow TM$ with $\pi \circ \sigma =$ identity and induces a metric g_Q in Q ([2], [6]). The transverse Riemannian connection D is a connection in Q that is torsionfree and metrical with respect to g_Q ([2], [6], [9]).

In a flat chart $U(x^i, x^\alpha)$ with respect to $\mathcal{F}$ ([8]), a local frame field $\{X_i, X_\alpha\} = \{\partial/\partial x^i, \partial/\partial x^\alpha - A^j{}_\alpha \partial/\partial x^j\}$ is called the basic adapted frame field of $\mathcal{F}$ ([6], [8], [10]). Here $A^j{}_\alpha$ are functions on U with $g(X_i, X_\alpha)=0$. It is trivial that $\{X_i\}$ (resp. $\{X_\alpha\}$) spans $\Gamma(E|_U)$ (resp. $\Gamma(E^\perp|_U)$), where $E^\perp = \sigma(Q)$ denotes the orthogonal complement bundle of E in TM. Hereafter, we omit the term "$|_U$" for simplicity. We set that $g_{ij}=g(X_i, X_j)$, $g_{\alpha\beta}=g(X_\alpha, X_\beta)$, $(g^{ij})=(g_{ij})^{-1}$, and $(g^{\alpha\beta})=(g_{\alpha\beta})^{-1}$.

Then we have

Lemma 1 ([10]). *It holds that* $D_{X_i}\pi(X_\alpha)=0$, $D_{X_i}D_{X_\alpha}\pi(X_\beta)=0$, *and* $D_{X_\alpha}\pi(X_\beta)=D_{X_\beta}\pi(X_\alpha)$.

Lemma 2 ([10]). *It holds that* $[X_\alpha, X_\beta] \in \Gamma(E)$.

The curvature R_D of D is defined by $R_D(X, Y)t = D_X D_Y t - D_Y D_X t - D_{[X,Y]}t$ for any X, $Y \in \Gamma(TM)$ and $t \in \Gamma(Q)$. We notice that $i(X)R_D = 0$, where $i(X)$ denotes the interior product with respect to $X \in \Gamma(E)$. Thus, for any $u, v \in \Gamma(Q)$, the operator $R_D(u, v): \Gamma(Q) \longrightarrow \Gamma(Q)$ is a well-defined endomorphism ([2]), that is, $R_D(u, v)t = D_{\sigma(u)}D_{\sigma(v)}t - D_{\sigma(v)}D_{\sigma(u)}t - D_{[\sigma(u), \sigma(v)]}t$. The Ricci operator ρ_D of $\mathcal{F}$ is given by $\rho_D(t) = g^{\alpha\beta}R_D(t, \pi(X_\alpha)) \cdot \pi(X_\beta)$ ([2]), and the transverse Ricci field

Ric_D of $\mathcal{F}$ is defined by

$$\mathrm{Ric}_D(t, u) = g_Q(\rho_D(t), u)$$

for any $t, u \in \Gamma(Q)$.

We set $V(\mathcal{F}) = \{Y \in \Gamma(TM) \mid [Y, X] \in \Gamma(E)$ for any $X \in \Gamma(E)\}$. An element of $V(\mathcal{F})$ is called an infinitesimal automorphism of $\mathcal{F}$([2]). We set $\bar{V}(\mathcal{F}) = \{s \in \Gamma(Q) \mid s = \pi(Y)$ and $Y \in V(\mathcal{F})\}$. The transverse Lie differentiation $\Theta(s)$ with respect to $s = \pi(Y) \in \bar{V}(\mathcal{F})$ is given by

$$\Theta(s)t = \pi([Y, Y_t])$$

for any $t \in \Gamma(Q)$ with $\pi(Y_t) = t$ and $Y_t \in \Gamma(TM)$ ([2], [4]). The transverse divergence $\mathrm{div}_D t$ of $t \in \Gamma(Q)$ with respect to D is given by $\mathrm{div}_D t = g^{\alpha\beta} \cdot g_Q(D_{X_\alpha} t, \pi(X_\beta))$, and the transverse gradient $\mathrm{grad}_D f$ of a function f is given by $\mathrm{grad}_D f = g^{\alpha\beta} X_\alpha(f) \cdot \pi(X_\beta)$ ([3], [6]).

For any $s = \pi(Y) \in \bar{V}(\mathcal{F})$, we have an operator $A_D(s) : \Gamma(Q) \longrightarrow \Gamma(Q)$ defined by $A_D(s) = \Theta(s) - D_Y$([2]).

Definition ([2], [4], [5]). If $s \in \bar{V}(\mathcal{F})$ satisfies $\Theta(s) g_Q = 2 f_s \cdot g_Q$ where f_s is a function on M, then s is called a transverse conformal field (t. c. f.) of $\mathcal{F}$ and f_s is called the characteristic function of s. If $s \in \bar{V}(\mathcal{F})$ satisfies $\Theta(s) g_Q = 0$, then s is called a transverse Killing field (t. K. f.) of $\mathcal{F}$. If $s \in \bar{V}(\mathcal{F})$ satisfies $(\Theta(s)D)_X t = \varphi_s(X) \cdot t + \varphi_s(\sigma(t)) \cdot \pi(X)$ for any $X \in \Gamma(TM)$ and $t \in \Gamma(Q)$, where φ_s is a 1-form on M, then s is a transverse projective field (t. p. f.) of $\mathcal{F}$ and φ_s is called the characteristic form of s. If $s \in \bar{V}(\mathcal{F})$ satisfies $\Theta(s) D = 0$, then s is called a transverse affine field (t. a. f.) of $\mathcal{F}$.

Then we have

Lemma 3. *$((\Theta(s)D)_{[X_\gamma, X_\alpha]}\pi(X_\beta) = 0$ for any $s \in \bar{V}(\mathcal{F})$.*

Proposition 1 ([6]). *If s is a t. c. f. of $\mathcal{F}$ with characteristic function f_s, then $\mathrm{div}_D s$ is a foliated function on M(i. e., $\mathrm{div}_D s$ has constant values on leaves) and $\mathrm{div}_D s = q \cdot f_s$.*

Proposition 2 ([6]). *If s is a t. p. f. of $\mathcal{F}$ with characteristic form φ_s, then $\mathrm{div}_D s$ is a foliated function on M, $d(\mathrm{div}_D s) = (q+1)\varphi_s$, and $\varphi_s(X) = 0$ for any $X \in \Gamma(E)$.*

Proposition 3 ([6]). *If s is a t. c. f. of $\mathcal{F}$ with characteristic function*

f_s, then it holds that

$$(\Theta(s)D)_{X_\alpha}\pi(X_\beta)=\{X_\alpha(f_s)\delta^\varepsilon_\beta+X_\beta(f_s)\delta^\varepsilon_\alpha-X_\gamma(f_s)g_{\alpha\beta}g^{\gamma\varepsilon}\}\cdot\pi(X_\varepsilon)$$

where δ^ε_β denotes the Kronecker's delta.

Let ∇ be the Levi-Civita connection with respect to g. The foliation $\mathcal{F}$ is harmonic if $g^{ij}\pi(\nabla_{X_i}X_j)=0$, that is, $\tau=-g^{ij}(D_{X_i}\pi)(X_j)=0$ ([2], [3]). We notice that $H=g^{ij}(\nabla_{X_i}X_j)_{E^\perp}=g^{ij}\cdot\{E^\perp\text{-component of }\nabla_{X_i}X_j\}$ is called the mean curvature vector field of each leaf of $\mathcal{F}$. Thus harmonic foliation $\mathcal{F}$ means that all leaves of $\mathcal{F}$ are minimal submanifolds M. Then we have

THEOREM 1 ([6], [11]). *Let $(M, g, \mathcal{F})$ be as above. If $\mathcal{F}$ is harmonic, then it holds that*

$$\int_M div_D t\, dM=0$$

for any $t\in\Gamma(Q)$.

THEOREM 2 ([6]). *Let $(M, g, \mathcal{F})$ be as above. If $\mathcal{F}$ is harmonic, then every t.a.f. of $\mathcal{F}$ is a t.K.f. of $\mathcal{F}$.*

THEOREM 3 ([7], [9]). *Let $(M, g, \mathcal{F})$ be as above, and let Δ be the Laplace-Beltrami operator acting on functions on M. Then it holds the following decomposition of Δ: $\Delta f=\Box' f+\Box''_0 f+H(f)$, for any function f on M.*

In Theorem 3, an operator $\Box'$ is defined by $\Box' f=-g^{ij}X_i(X_j(f))+g^{ij}(\nabla_{X_i}X_j)_E(f)$, where $(\ )_E$ denotes the E-component of $(\ )$. If f is a foliated function on M, then we have that $\Delta f=\Box''_0 f+H(f)$, where $\Box''_0 f=-g^{\alpha\beta}X_\alpha(X_\beta(f))+g^{\alpha\beta}(\nabla_{X_\alpha}X_\beta)_{E^\perp}(f)$.

3. Proof of Theorem A

Let $s\in\bar{V}(\mathcal{F})$ be a t.c.f. of $\mathcal{F}$ with characteristic function f_s. For any $t, u\in\Gamma(Q)$, we have

$$(\Theta(s)\mathrm{Ric}_D)(t,u)=2f_s\cdot g_Q(\rho_D(t),u)+g_Q((\Theta(s)\rho_D)(t),u)$$
$$(\Theta(s)\rho_D)(t)=-2f_s\cdot\rho_D(t)+g^{\alpha\beta}(\Theta(s)R_D)(t,\pi(X_\alpha))\pi(X_\beta).$$

Thus we have

$$(\#)\quad(\Theta(s)\mathrm{Ric}_D)(t,u)=g^{\alpha\beta}g_Q((\Theta(s)R_D)(t,\pi(X_\alpha))\pi(X_\beta),u).$$

We calculate $((\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta)$ using Lemmas and Propositions in the above section. Then we have

$$\begin{aligned}
&((\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta)\\
&=-X_\alpha(X_\beta(f_s))\cdot\pi(X_\gamma)+(\nabla_{X_\alpha}X_\beta)_{E^\perp}(f_s)\cdot\pi(X_\gamma)\\
&\quad -g_{\alpha\beta}\cdot D_{X_\gamma}\,\mathrm{grad}_D f_s+g_{\gamma\beta}\cdot D_{X_\alpha}\,\mathrm{grad}_D f_s+X_\gamma(X_\beta(f_s))\cdot\pi(X_\alpha)\\
&\quad -(\nabla_{X_\gamma}X_\beta)_{E^\perp}(f_s)\cdot\pi(X_\alpha),
\end{aligned}$$

and, since $g^{\alpha\beta}X_\gamma(X_\beta(f_s))\cdot\pi(X_\alpha)-g^{\alpha\beta}(\nabla_{X_\gamma}X_\beta)_{E^\perp}(f_s)\cdot\pi({}^\alpha X_\alpha)=D_{X_\gamma}$ $\mathrm{grad}_D f_s$, we have

$$\begin{aligned}
&g^{\alpha\beta}(\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta)\\
&=\{-g^{\alpha\beta}X_\alpha(X_\beta(f_s))+g^{\alpha\beta}(\nabla_{X_\alpha}X_\beta)_{E^\perp}(f_s)\}\cdot\pi(X_\gamma)\\
&\quad -(q-2)\cdot D_{X_\gamma}\,\mathrm{grad}_D f_s\\
&=\square''_0 f_s\cdot\pi(X_\gamma)-(q-2)\cdot D_{X_\gamma}\,\mathrm{grad}_D f_s.
\end{aligned}$$

By (♯), we have

$$\begin{aligned}
&(\Theta(s)\mathrm{Ric}_D)(\pi(X_\gamma),\pi(X_\varepsilon))\\
&=g_Q(g^{\alpha\beta}(\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta),\pi(X_\varepsilon))\\
&=g_{\gamma\varepsilon}\cdot\square''_0 f_s-(q-2)\cdot g_Q(D_{X_\gamma}\,\mathrm{grad}_D f_s,\pi(X_\varepsilon)).
\end{aligned}$$

Since $g^{\gamma\varepsilon}g_Q(D_{X_\gamma}\ \mathrm{grad}_D f_s,\pi(X_\varepsilon))=-\square''_0 f_s$, we have

$$g^{\gamma\varepsilon}(\Theta(s)\mathrm{Ric}_D)(\pi(X_\gamma),\pi(X_\varepsilon))=2(q-1)\cdot\square''_0 f_s.$$

By our assumption: $\Theta(s)\mathrm{Ric}_D=0$ and $q\geq 2$, we have that $\square''_0 f_s=0$. Since $\mathcal{F}$ is harmonic (i.e., $H=0$) and f_s is a foliated function on M, by Theorem 3, we have $\Delta f_s=0$ so that $f_s=$const. on M. Proposition 1 and Theorem 1 imply that f_s vanishes identically. Thus s is a t. K. f. of $\mathcal{F}$.

4. Proof of Theorem B

For any $s=\pi(Y)\in\bar{V}(\mathcal{F})$, we have

$$\begin{aligned}
&(\Theta(s)\mathrm{Ric}_D)(\pi(X_\gamma),\pi(X_\varepsilon))\\
&=Y(g_Q(\rho_D(\pi(X_\gamma)),\pi(X_\varepsilon)))-g_Q(\rho_D(\Theta(s)\pi(X_\gamma)),\pi(X_\varepsilon))\\
&\qquad\qquad -g_Q(\rho_D(\pi(X_\gamma)),\Theta(s)\pi(X_\varepsilon))\\
&=g_Q((D_Y\rho_D)(\pi(X_\gamma)),\pi(X_\varepsilon))+g_Q(\rho_D(D_Y\pi(X_\gamma)),\pi(X_\varepsilon))\\
&\quad +g_Q(\rho_D(\pi(X_\gamma)),D_Y\pi(X_\varepsilon))-g_Q(\rho_D(\Theta(s)\pi(X_\gamma)),\pi(X_\varepsilon))\\
&\quad -g_Q(\rho_D(\pi(X_\gamma)),\Theta(s)\pi(X_\varepsilon))\\
&=g_Q((\Theta(s)\rho_D)(\pi(X_\gamma)),\pi(X_\varepsilon))-g_Q((A_D(s)\rho_D)(\pi(X_\gamma)),\pi(X_\varepsilon))\\
&\quad -g_Q(\rho_D(A_D(s)\pi(X_\gamma)),\pi(X_\varepsilon))-g_Q(\rho_D(\pi(X_\gamma)),A_D(s)\pi(X_\varepsilon)).
\end{aligned}$$

We have

$$(\Theta(s)\rho_D)(\pi(X_\gamma))$$

$$
\begin{aligned}
&= g^{\alpha\beta}(\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta)\\
&\quad + g^{\alpha\beta}R_D(\pi(X_\gamma),A_D(s)\pi(X_\alpha))\pi(X_\beta)\\
&\quad + g^{\alpha\beta}R_D(\pi(X_\gamma),\pi(X_\alpha))A_D(s)\pi(X_\beta),
\end{aligned}
$$

and

$$
\begin{aligned}
&g^{\gamma\varepsilon}g_Q((\Theta(s)\rho_D)(\pi(X_\gamma)),\pi(X_\varepsilon))\\
&= g^{\gamma\varepsilon}g^{\alpha\beta}g_Q((\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta),\pi(X_\varepsilon))\\
&\quad + g^{\gamma\varepsilon}g_Q(\rho_D(A_D(s)\pi(X_\gamma)),\pi(X_\varepsilon))\\
&\quad + g^{\gamma\varepsilon}g_Q(\rho_D(\pi(X_\gamma)),A_D(s)\pi(X_\varepsilon)).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&g^{\gamma\varepsilon}(\Theta(s)\mathrm{Ric}_D)(\pi(X_\gamma),\pi(X_\varepsilon))\\
&= g^{\gamma\varepsilon}g^{\alpha\beta}g_Q((\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta),\pi(X_\varepsilon))\\
&\quad - g^{\gamma\varepsilon}g_Q((A_D(s)\rho_D)(\pi(X_\gamma)),\pi(X_\varepsilon)).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&g^{\gamma\varepsilon}g_Q((A_D(s)\rho_D)(\pi(X_\gamma)),\pi(X_\varepsilon))\\
&= g^{\gamma\varepsilon}g_Q(A_D(s)(\rho_D(\pi(X_\gamma)),\pi(X_\varepsilon))\\
&\quad - g^{\gamma\varepsilon}g_Q(\rho_D(A_D(s)\pi(X_\gamma)),\pi(X_\varepsilon))\\
&= g^{\gamma\varepsilon}g_Q(\rho_D(\pi(X_\gamma)),{}^tA_D(s)\pi(X_\varepsilon))\\
&\quad - g^{\gamma\varepsilon}g_Q(A_D(s)\pi(X_\gamma),\rho_D(\pi(X_\varepsilon)))\\
&= -g^{\gamma\varepsilon}g_Q(\rho_D(\pi(X_\gamma)),(A_D(s)-{}^tA_D(s))\pi(X_\varepsilon))\\
&= 0
\end{aligned}
$$

because the operator ρ_D (resp. $A_D(s)-{}^tA_D(s)$) is symmetric (resp. skew-symmetric) with respect to g_Q.

Thus we have

(##)
$$
\begin{aligned}
&g^{\gamma\varepsilon}(\Theta(s)\mathrm{Ric}_D)(\pi(X_\gamma),\pi(X_\varepsilon))\\
&= g^{\gamma\varepsilon}g^{\alpha\beta}g_Q((\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta),\pi(X_\varepsilon)).
\end{aligned}
$$

Let $s=\pi(Y)\in\overline{V}(\mathcal{F})$ be a t. p. f. of $\mathcal{F}$ with the characteristic form φ_s. Then, by Lemmas and Propositions insection 2, we have

$$
\begin{aligned}
&(\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta)\\
&= (\Theta(s)D)_{X_\gamma}(D_{X_\alpha}\pi(X_\beta)) + D_{X_\gamma}((\Theta(s)D)_{X_\alpha}\pi(X_\beta))\\
&\quad - (\Theta(s)D)_{X_\alpha}(D_{X_\gamma}\pi(X_\beta)) - D_{X_\alpha}((\Theta(s)D)_{X_\gamma}\pi(X_\beta))\\
&= \varphi_s(X_\gamma)\cdot(D_{X_\alpha}\pi(X_\beta)) + \varphi_s(\sigma(D_{X_\alpha}\pi(X_\beta)))\cdot\pi(X_\gamma)\\
&\quad + D_{X_\gamma}\{\varphi_s(X_\alpha)\cdot\pi(X_\beta)+\varphi_s(X_\beta)\cdot\pi(X_\alpha)\}\\
&\quad - \varphi_s(X_\alpha)\cdot(D_{X_\gamma}\pi(X_\beta)) - \varphi_s(\sigma(D_{X_\gamma}\pi(X_\beta)))\cdot\pi(X_\alpha)\\
&\quad - D_{X_\alpha}\{\varphi_s(X_\gamma)\cdot\pi(X_\beta)+\varphi_s(X_\beta)\cdot\pi(X_\gamma)\}\\
&= (D_{X_\gamma}\varphi_s)(X_\alpha)\cdot\pi(X_\beta) + (D_{X_\gamma}\varphi_s)(X_\beta)\cdot\pi(X_\alpha)\\
&\quad - (D_{X_\alpha}\varphi_s)(X_\gamma)\cdot\pi(X_\beta) - (D_{X_\alpha}\varphi_s)(X_\beta)\cdot\pi(X_\gamma).
\end{aligned}
$$

Thus we have

$$\begin{aligned} &g^{\gamma\varepsilon}g^{\alpha\beta}g_Q((\Theta(s)R_D)(\pi(X_\gamma),\pi(X_\alpha))\pi(X_\beta),\pi(X_\varepsilon))\\ &=-(q-1)g^{\gamma\varepsilon}(D_{X_\gamma}\varphi_s)(X_\varepsilon)\\ &=(q-1)\{\delta\varphi_s-\varphi_s(H)\},\end{aligned}$$

where δ denotes the adjoint operator of d.

By (##), we have

$$g^{\gamma\varepsilon}(\Theta(s)\mathrm{Ric}_D)(\pi(X_\gamma),\pi(X_\varepsilon))=(q-1)\cdot\{\delta\varphi_s-\varphi_s(H)\}.$$

If $\mathcal{F}$ is harmonic, $q\geq 2$, and $\Theta(s)\mathrm{Ric}_D=0$, then we have that $\delta\varphi_s=0$. By Proposition 2, we have that $\delta d(\mathrm{div}_D s)=0$, which implies $\mathrm{div}_D s$ $=$constant. Again, by Propostion 2, we have that $\varphi_s=0$. Thus s is a t. a. f. of $\mathcal{F}$. By Theorem 2, s is a t. K. f. of $\mathcal{F}$.

Addendum of the preceding paper [6] appeared in J. Korean Math. Soc. 25 (1988).

First of all, we have to correct Theorem 2.7 in [6, pp. 88]: (iv) in Theorem 2.7 should be read that if s is a t. p. f. of $\mathcal{F}$ then

$$(*)\qquad \Delta_D s=D_{\sigma(\tau)}s+\rho_D(s)-\frac{2}{q+1}\mathrm{grad}_D\ \mathrm{div}_D s.$$

This is a printing mistake.

Next, we will give a proof of the following Theorem:

THEOREM C. *Let $(M, g, \mathcal{F})$ be as Theorem A, and s be a t. p. f. of $\mathcal{F}$. Suppose that $\mathcal{F}$ is harmonic. If ρ_D is non-positive everywhere and negative for at least one point of M, then $s=0$.*

Proof of Theorem C. Since we have that $\mathrm{Tr}(A_D(s)A_D(s))=\mathrm{Tr}({}^tA_D(s)A_D(s))+\frac{1}{2}\cdot\mathrm{Tr}((A_D(s)-{}^tA_D(s))^2)$ and, by Theorem 4.1 in [6], $\int_M\{g_Q(\rho_D(s),s)+\mathrm{Tr}(A_D(s)A_D(s))-(\mathrm{div}_D s)^2\}dM=0$, we have

$$\begin{aligned}&\langle\langle\rho_D(s),s\rangle\rangle+\frac{1}{2}\int_M\mathrm{Tr}((A_D(s)-{}^tA_D(s))^2)dM\\ &+\int_M\mathrm{Tr}({}^tA_D(s)A_D(s))dM-\int_M(\mathrm{div}_D s)^2\,dM=0.\end{aligned}$$

Here, $\langle\langle u,v\rangle\rangle=\int_M g_Q(u,v)dM$ for any $u,v\in\Gamma(Q)$. By the fact that $\int_M\mathrm{Tr}({}^tA_D(s)A_D(s))\,dM=\langle\langle Ds,Ds\rangle\rangle$ and $\langle\langle\Delta_D s,s\rangle\rangle=\langle\langle Ds,Ds\rangle\rangle$ ([6]), we have

$$\langle\!\langle \rho_D(s), s\rangle\!\rangle + \frac{1}{2}\int_M \mathrm{Tr}((A_D(s) - {}^tA_D(s))^2)dM$$
$$+\langle\!\langle \varDelta_D s, s\rangle\!\rangle - \int_M (\mathrm{div}_D s)^2\ dM = 0.$$

Since s is a t. p. f. of $\mathcal{F}$ and $\mathcal{F}$ is harmonic, by the equalities: $\langle\!\langle \mathrm{grad}_D \mathrm{div}_D s, s\rangle\!\rangle = -\int_M (\mathrm{div}_D s)^2\ dM$ ((3. 3) in [6]) and (*), we have

$$\begin{aligned} 0 &= \langle\!\langle \rho_D(s), s\rangle\!\rangle + \frac{1}{2}\int_M \mathrm{Tr}((A_D(s) - {}^tA_D(s))^2)dM \\ &\quad + \langle\!\langle \varDelta_D s, s\rangle\!\rangle - \int_M (\mathrm{div}_D s)^2 dM \\ &= 2\langle\!\langle \rho_D(s), s\rangle\!\rangle + \frac{1}{2}\int_M \mathrm{Tr}((A_D(s) - {}^tA_D(s))^2)dM \\ &\quad - \frac{q-1}{q+1}\int_M (\mathrm{div}_D s)^2 dM. \end{aligned}$$

By the assumption of ρ_D, we have that $\langle\!\langle \rho_D(s), s\rangle\!\rangle \leq 0$. Since $A_D(s) - {}^tA_D(s)$ is a skew-symmetric operator, we have that $\int_M \mathrm{Tr}((A_D(s) - {}^tA_D(s))^2)dM \leq 0$. Thus we have that $\langle\!\langle \rho_D(s), s\rangle\!\rangle = 0$. Therefore, we have that $s=0$.

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