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THE GENERALIZED BESSEL TRANSFORMATIONS ON THE SPACES $L'_{p,\nu}$ OF DISTRIBUTIONS

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1. Introduction

Two variants of the Hankel transformations that are called Bessel transformations (they are also called Hankel–Schwartz's transformations) are defined by

$$B_{\mu,1}(f)(y)=\int_0^\infty x^{2\mu+1}b_\mu(xy)f(x)\,dx$$

and

$$B_{\mu,2}(f)(y)=y^{2\mu+1}\int_0^\infty b_\mu(xy)f(x)\,dx.$$

Here $b_\mu(z)=z^{-\mu}J_\mu(z)$, where J_μ is the Bessel function of the first kind and order μ. These transformations have been studied extensively in the last years. Significant papers are the ones of G. Altenburg [1], A. Schuitman [13], W.Y. Lee [5], A.M. Sanchez [12] and J.M. Mendez [6], amongst others.

In this paper we study the behaviour of the Bessel transformations on the spaces $L_{p,\nu}$ introduced by P.G. Rooney [9]. We prove that the $B^{\mu,1}$-transformation is a bounded linear operator of $L_{p,\nu}$ into $L_{p,p(2\mu+2)-\nu}$, provided that $1<p<\infty$, $\mu>-\frac{1}{2}$ and $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$. Also, if $1<p<\infty$, $\mu>-\frac{1}{2}$ and $-\mu+\frac{1}{2}<\frac{\nu}{p}<1$ then $B_{\mu,2}$ is a bounded linear operator of $L_{p,\nu}$ into $L_{p,-\nu-2\mu p}$.

Moreover if $f\in L_{q,(\nu-2\mu-1)q-\nu}$ and $g\in L_{p,\nu}$ then

$$\int_0^\infty f(x)B_{\mu,1}(g)(x)\,dx=\int_0^\infty B_{\mu,2}(f)(y)g(y)\,dy \qquad (1)$$

provided that $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\mu>-\frac{1}{2}$ and $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$.

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The mixed Parseval's equation (1) suggests definitions of the generalized Bessel transformations. More exactly, we define the generalized $B'_{\mu,1}$-transformation on $L'_{q,\nu+(1-\nu)q}$ as the adjoint of the classical $B_{\mu,2}$-transform, so that

$$<B'_{\mu,1}f, \phi> = <f, B_{\mu,2}\phi>, \qquad (2)$$

for every $f \in L'_{q,\nu+(1-\nu)q}$ and for every $\phi \in L_{q,(\nu-2\mu-1)q-\nu}$, provided that $1<q<\infty$, $\mu>-\frac{1}{2}$ and $\mu+\frac{3}{2}<\nu\left(1-\frac{1}{q}\right)<2\mu+2$.

The generalized $B'_{\mu,2}$-transformation on $L'_{p,(2\mu+2)p-\nu}$ is defined as the adjoint of the $B_{\mu,1}$-transformation, through

$$<B'_{\mu,2}f, \phi> = <f, B_{\mu,1}\phi>, \quad \forall f \in L'_{p,(2\mu+2)p-\nu}, \quad \forall \phi \in L_{p,\nu} \qquad (3)$$

provided that $1<p<\infty$, $\mu>-\frac{1}{2}$ and $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$.

Note that (2) and (3) appear to be extensions of the mixed Parseval equation (1).

We now briefly recall definitions and some properties of space $L_{p,\nu}$ introduced by P. G. Rooney [9]. Suppose $1\leq p<\infty$, ν is real, and denote by $L_{p,\nu}$ the collection of functiions f, measurable on $(0,\infty)$, and which satisfy

$$\|f\|_{p,\nu} = \left[\int_0^\infty x^{\nu-1}|f(x)|^p dx\right]^{1/p} < \infty.$$

The space $D(0,\infty)$ consists of all smooth complex-valued function having compact support contained in $(0,\infty)$.

Proposition 1. *$D(0,\infty)$ is dense in $L_{p,\nu}$ for any ν and any p satisfying $1\leq p<\infty$.*

For $\gamma>0$, $Re\,\alpha>0$, $Re\,\beta>0$, let

$$(I_{\gamma,\alpha,\zeta}f)(x) = \frac{\nu}{\Gamma(\alpha)}\int_0^1 (1-u^\gamma)^{\alpha-1}u^{\nu\zeta-1}f(ux)\,du \qquad (4)$$

$$(J_{\gamma,\beta,\eta}f)(x) = \frac{\nu}{\Gamma(\alpha)}\int_1^\infty (u^\gamma-1)^{\alpha-1}u^{-\nu(\beta+\eta-1)-1}f(xu)\,du \qquad (5)$$

where ζ and η are complex numbers. $I_{\nu,\alpha,\zeta}$ and $J_{\nu,\beta,\eta}$ are generalizations of the Riemann–Liouville and Weyl fractional integrals, respectively. There are vast literatures of these fractional integrals, particularly for $\nu=1$ and $\nu=2$; see [4] for an excellent summary, and [3] for many applications.

A property that will be useful in the sequel is the following one.

PROPOSITION 2. (P.G. Rooney [9]). *If* $\frac{\gamma}{p\nu}<Re\,\zeta$, $I_{\nu,\alpha,\zeta}$ *is a bounded linear operator of* $L_{p,\gamma}$ *into itself and if* $\frac{\gamma}{p\nu}\gg -Re\,\eta$, *then* $J_{\nu,\beta,\eta}$ *is a bounded linear operator of* $L_{p,\gamma}$ *into itself.*

The following behaviours near the origin and the infinity of the function $b_\mu(z)$ can be deduced of the correspondent ones of the Bessel function $J_\mu(z)$ and they will be used in the sequel,

$$b_\mu(z)=O(1),\ \text{as } z\to 0 \tag{6}$$

$$b_\mu(z)\simeq z^{-\mu-(1/2)},\ \text{as } z\to\infty \tag{7}$$

2. The Bessel transformations on the spaces $L_{p,\nu}$

In this section we study the behaviours of the Bessel transformations on the spaces $L_{p,\nu}$.

THEOREM 1. *The* $B_{\mu,1}$*-transformation is a bounded linear operator of* $L_{p,\nu}$ *into* $L_{p,p(2\mu+2)-\nu}$, *provided that* $1\leq p<\infty$, $\mu>-\frac{1}{2}$ *and* $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$.

Proof. According to the definition of the $B_{\mu,1}$-transform, we have

$$B_{\mu,1}(\phi)\left(\frac{1}{y}\right)=\int_0^\infty b_\mu\left(\frac{x}{y}\right)x^{2\mu+1}\phi(x)\,dx=y^{2\mu+2}\int_0^\infty b_\mu(u)\,u^{2\mu+1}\phi(uy)\,du$$

In virtue of behaviours (6) and (7) of the function b_μ, it follows

$$\left|y^{-2\mu-2}B_{\mu,1}(\phi)\left(\frac{1}{y}\right)\right|$$

$$\leq C\left[\int_0^1 u^{2\mu+1}|\phi(uy)|\,du+\int_1^\infty u^{\mu+(1/2)}|\phi(uy)|\,du\right] \tag{8}$$

for a certain positive constant C and $\phi\in L_{p,\nu}$.

According to (4) and (5) we can write

$$(I_{1,1,2\mu+2}\phi)(y)=\int_0^1 u^{2\mu+1}\phi(uy)\,du$$

$$(J_{1,1,-\mu-(9/2)}\phi)(y)=\int_1^\infty u^{\mu+(1/2)}\phi(uy)\,du$$

and by using Proposition 2, $I_{1,1,2\mu+2}$ and $J_{1,1,-\mu-(3/2)}$ are bounded linear operators of $L_{p,\nu}$ into itself, provided that $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$, $1<p<\infty$ and $\mu>-\frac{1}{2}$.

Then, from (8) if $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$, $1<p<\infty$ and $\mu>-\frac{1}{2}$,

$$\left\| y^{-2\mu-2}B_{\mu,1}(\phi)\left(\frac{1}{y}\right)\right\|_{\nu,p}\leq C_1\|\phi\|_{\nu,p}$$

where C_1 is a positive constant; and in other words,

$$\|B_{\mu,1}(\phi)(y)\|_{p(2\mu+2)-\nu,p}\leq C_1\|\phi\|_{\nu,p}$$

under above conditions.

Hence, $B_{\mu,1}$ is a bounded linear operator of $L_{p,\nu}$ into $L_{p,p(2\mu+2)-\nu}$ under the imposed hypotheses.

On the other hand, since

$$B_{\mu,2}(\phi)(y)=y^{2\mu+1}B_{\mu,1}(x^{-2\mu-1}\phi)(y)$$

we can deduce from Theorem 1, the following

Corollary 1. *If* $1<p<\infty$, $\mu>-\frac{1}{2}$ *and* $-\mu+\frac{1}{2}<\frac{\nu}{p}<1$, *then* $B_{\mu,2}$ *is a bounded linear operator from* $L_{p,\nu}$ *into* $L_{p,-\nu-2\mu p}$.

Remark. The behaviour of the Hankel transformation on the spaces $L_{p,\nu}$ was studied by P.G. Rooney in [10] and [11]. However the approach followed by Rooney is essentially different to the method used by us in this paper.

3. The generalized Bessel transformations on $L'_{p,\nu}$

This section is devoted to define the generalized Bessel transformations on the spaces $L'_{p,\nu}$ of distributions. Definitions of said transformations can be understood as generalizations of the mixed Parseval's equation that is proved in the next

Proposition 3. *If* $f\in L_{p,\nu}$, $g\in L_{q,(\nu-2\mu-1)q-\nu}$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\mu>-\frac{1}{2}$ *and* $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$, *then*

$$\int_0^\infty g(x)B_{\mu,1}(f)(x)dx=\int_0^\infty B_{\mu,2}(g)(y)f(y)dy \qquad (9)$$

Proof. If $f\in D(0,\infty)$ and $g\in D(0,\infty)$, then

$$\int_0^\infty g(x)B_{\mu,1}(f)(x)dx=\int_0^\infty g(x)\int_0^\infty y^{2\mu+1}b_\mu(xy)f(y)dydx$$
$$=\int_0^\infty f(y)y^{2\mu+1}\int_0^\infty b_\mu(xy)g(x)dxdy=\int_0^\infty B_{\mu,2}(g)(y)f(y)dy$$

the interchange of the orders of the integration being easily justified by Fubini's theorem. Thus (9) is true if $f\in D(0,\infty)$ and $g\in D(0,\infty)$, and hence, since $D(0,\infty)$ is dense in $L_{p,\nu}$, the general result will be true if we show that both sides of (9) represent bounded bilinear functionals on $L_{p,\nu}\times L_{q,q(2\mu+2)-\nu}$.

Now, since $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, by using Holder's inequality, one has

$$\left|\int_0^\infty g(x)B_{\mu,1}f(x)dx\right|\leq\|B_{\mu,1}(f)\|_{p(2\mu+2)-\nu,p}\|g\|_{(\nu-2\mu-1)q-\nu,q}$$

Moreover, since $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$, $B_{\mu,1}$ is a bounded linear operator of $L_{p,\nu}$ into $L_{p,p(2\mu+2)-\nu}$, and we can deduce

$$\left|\int_0^\infty g(x)B_{\mu,1}(f)(x)dx\right|\leq C\|f\|_{\nu,p}\|g\|_{(\nu-2\mu-1)q-\nu,q}$$

where C is a certain positive constant, so that the left hand side of (9) is a bounded bilinear function on $L_{p,\nu}\times L_{q,q(2\mu+2)-\nu}$, as the right hand side on (9) by a similar calculation, and the result follows.

We now define the generalized $B'_{\mu,1}$-transformation on $L'_{q,\nu+(1-\nu)q}$ as the adjoint on the classical $B_{\mu,2}$-transform, through

$$\langle B'_{\mu,1}f,\phi\rangle=\langle f,B_{\mu,2}\phi\rangle,\quad f\in L'_{q,\nu+(1-\nu)q},\ \phi\in L_{q,(\nu-2\mu-1)q-\nu} \qquad (10)$$

From Corollary 1, we can deduce the following

THEOREM 2. *If* $1<q<\infty$, $\mu>-\frac{1}{2}$ *and* $\mu+\frac{3}{2}<\nu\left(1-\frac{1}{q}\right)<2\mu+2$ *then the generalized* $B'_{\mu,1}$*-transformation is a bounded linear operator of* $L'_{q,\nu+(1-\nu)q}$ *into* $L'_{q,(\nu-2\mu-1)q-\nu}$.

Note that definition (10) represents an extension of the mixed Parseval's equation (9).

On the other hand, if $f\in L_{p,\nu}$ then by invoking again Holder's inequality

$$\left|\int_0^\infty f(x)\phi(x)dx\right| \leq \|f\|_{\nu,p}\|\phi\|_{\nu+(1-\nu)q,q}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Hence f generates a regular distribution in $L'_{q,\nu+(1-\nu)q}$ and, in this sense, $L_{p,\nu}$ is contained in $L'_{q,\nu+(1-\nu)q}$.

Thus if $f\in L_{p,\nu}$ then we can define the generalized $B'_{\mu,1}$-transformation $B'_{\mu,1}f$ of f and the classical $B_{\mu,1}$-transformation $B_{\mu,1}f$ of f. We can prove that $B'_{\mu,1}f=B_{\mu,1}f$ in the sense of equality in $L'_{q,(\nu-2\mu-1)q-\nu}$. In effect, if $\phi\in L_{q,(\nu-2\mu-1)q-\nu}$ then by using (9)

$$\langle B_{\mu,1}f,\phi\rangle=\int_0^\infty (B_{\mu,1}f)(x)\phi(x)dx=\int_0^\infty f(x)(B_{\mu,2}f)(x)dx$$
$$=\langle f, B_{\mu,2}f\rangle=\langle B'_{\mu,1}f,\phi\rangle$$

Therefore the classical $B_{\mu,1}$-transformation is a special case of the generalized $B'_{\mu,1}$-transform.

In a similar way we can define the $B'_{\mu,2}$-transformation. More exactly if $f\in L'_{p,p(2\mu+2)-\nu}$ the $B'_{\mu,2}$-transformation $B'_{\mu,2}f$ of f is defined by

$$\langle B'_{\mu,2}f,\phi\rangle=\langle f, B_{\mu,1}\phi\rangle,\ \text{for every } \phi\in L_{p,\nu}$$

and from Theorem 1 we can deduce

THEOREM 3. *If* $1<p<\infty$, $\mu>-\frac{1}{2}$ *and* $\mu+\frac{3}{2}<\frac{\nu}{p}<2\mu+2$, *then the generalized* $B'_{\mu,2}$*-transformation is a bounded linear operator of* $L'_{p,p(2\mu+2)-\nu}$ *into* $L'_{p,\nu}$.

We can see that the classical $B_{\mu,2}$-transformation is a special case of the generalized $B'_{\mu,2}$-transformation.

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