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CONTINUITIES OF THE EVALUATION MAPS OF LOCALLY CONVEX SPACES INTO THEIR BIDUALS

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1. Introduction

Let E be a locally convex topological vector space and (τ) denote a certain property of subsets in E. If the dual space E^* of E is endowed with the topology of uniform convergence on sets with property (τ), then E^* is also a locally convex topological vector space, which will be denoted by $E^{*\tau}$. In this note, we consider the question when the evaluation map $e: E \rightarrow E^{*\tau*\sigma}$ defined by $e(x)(f)=f(x)$ for $x \in E$ and $f \in E^*$, is a topological isomorphism, for the following cases:

(p); finite,

(c); convex compact,

(k); compact,

(t); totally bounded,

(b); bounded

We have already consider the same question in the cases $\tau=\sigma$ in [2]. Note that e is always injective, and we have already consider the question of surjectivity in [2] under the name 'condition $[\tau_2]$', (note that (σ) is irrelevant in this case). Such conditions are closedly related with the various 'completeness' conditions. Also, surjectivity of e implies openness in most cases (cf. Proposition 2.2). Hence, our main theme is the question of continuity of e, and we say that E satisfy the *condition* $[\tau\sigma]$ if the evaluation map $e: E \rightarrow E^{*\tau*\sigma}$ is continuous.

Such conditions $[\tau\sigma]$'s are all weaker than barrelledness, and may be considered as kinds of 'equicontinuous question' (cf. [1, p. 136]). In other word, they relate the concept of equicontinuity with compactness, total boundedness and boundedness with respect to the various topologies in the dual space. Actually, the condition $[pb]$ (respectively $[bb]$) is exactly the barrelledness (respectively quasi-barrelledness).

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Also, E is a Mackey space if and only if E satisfies the condition $[pc]$. In [2], we have already considered the conditions $[cc]$, $[kk]$, $[tt]$ under the names $[c_1]$, $[k_1]$, $[t_1]$ respectively. In this note, we investigate relations between various conditions $[\tau\sigma]$'s, and provide related examples of locally convex spaces. Throughout this note, we follow the notations and terminologies in [2], and E always denotes a locally convex topological vector space over the field of complex numbers.

2. Conditions $[\tau\sigma]$'s

The following two propositions are modifications of [2, Theorem 3. 1].

PROPOSITION 2. 1. *Let E be a locally convex topological vector space. Then the following are equivalent:*

i) *E satisfies the condition $[\tau\sigma]$.*

ii) *If K is a σ-set in $E^{*\tau}$ then the polar K_o of K is a neighborhood in E.*

iii) *every σ-set K in $E^{*\tau}$ is equicontinuous on E.*

Proof. Because e is injective, we have the relation $e^{-1}(K^o)=K_o$. For the equivalence of i) and ii), note that non-zero scalar multiples of polars K^o of subsets K in $E^{*\tau}$ with property σ forms a local base for the topology of $E^{*\tau*\sigma}$. The equivalence between ii) and iii) is trivial.

PROPOSITION 2. 2. *Let τ be one of p, c, k, t and σ be one of c, k, t, b. Then, surjectivity of e implies openness. Also, this is the case when $\tau=\sigma=b$.*

Proof. Note that $e(U)=U^{oo}$ *for every subset* U of E because e is surjective. Let U be a closed convex circled neighborhood of 0 in E. Then U^o is compact in E^t by [2, Theorem 2. 2], and so, it is compact in E^τ for $\tau=p, c, k, t$. Hence, U^o has property (σ) for $\sigma=c, k, t, b$, and $e(U)=U^{oo}$ is a neighborhood in $E^{*\tau*\sigma}$. For the case $\tau=\sigma=b$, note that compact convex set in E^{*t} is bounded in E^{*b}, and so, U^o is bounded in E^b.

From Proposition 2.1, the implications in Figure 1 are clear, and the conditions $[\tau p]$'s are always true. In order to apply Proposition 2.1 in the space E itself, we need to characterize the polars of a σ-sets in $E^{*\tau}$. In [2, Theorem 2.2], this was done in the cases of $\tau=k, t$ and $\sigma=c, k, t$, via the concepts of k-and t-neighborhoods. Recall that a subset U containing 0 in E is a τ-neighborhood if for every subset K with property (τ) there exists a neighborhood V of 0 such that $V\cap K\subseteq U\cap K$. The following easy lemma deals with the case $\sigma=b$. We say that a subset U is *τ-absorbing* if U absorbs every τ-set in E.

Lemma 2.3. *A subset U of E is the polar of a bounded set B in $E^{*\tau}$ if and only if U is a τ-absorbing barrel.*

$$\begin{array}{ccccccccc}
[pb] & \to & [cb] & \to & [kb] & \to & [tb] & \to & [bb] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[pt] & \to & [ct] & \to & [kt] & \to & [tt] & \to & [bt] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[pk] & \to & [ck] & \to & [kk] & \to & [tk] & \to & [bk] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[pc] & \to & [cc] & \to & [kc] & \to & [tc] & \to & [bc]
\end{array}$$

Figure 1

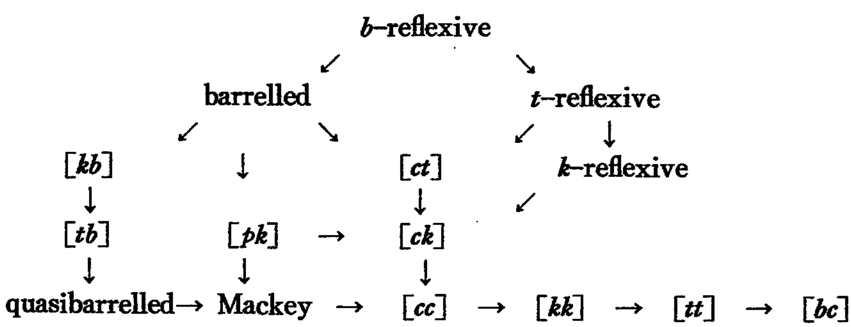

Figure 2

Proof. Let B be a bounded set in $E^{*\tau}$, and K be a τ-set in E. Then, B is absorbed by K°, and so, the polar B_o absorbs K. It is clear that B_o is a barrel. Conversely, if U is a τ-absorbing barrel and K be a τ-set, then U absorbs K and U° is absorbed by K°, a neighborhood in $E^{*\tau}$. Hence, U° is bounded in $E^{*\tau}$.

COROLLARY 2.4. *A locally convex space E satisfies the condition $[\tau b]$ if and only if every τ-absorbing barrel in E is a neighborhood.*

For the more implications and equivalences than in Figure 1, we have the following Proposition 2.5, and get implications in Figure 2. Also, we need several facts in Proposition 2.6 in the next section.

PROPOSITION 2.5. *We have the following*:
i) *Three conditions $[pb]$, $[pt]$ and $[cb]$ are all equivalent to barrelledness.*
ii) *The condition $[bb]$ (i.e. quasibarrelledness) implies $[pc]$ (i.e. being Mackey).*
iii) *Three conditions $[kt]$, $[kk]$ and $[kc]$ are equivalent each other.*
iv) *Three conditions $[tt]$, $[tk]$ and $[tc]$ are equivalent each other.*
v) *Three conditions $[bt]$, $[bk]$ and $[bc]$ are equivalent each other.*

Proof. i) For the relation $[pb]\leftrightarrow[pt]$, note that boundedness and total boundedness are equivalent in the weak*-topology (cf. [3, Exercise 8.2.6]). For the implication $[cb]\rightarrow[pb]$, note that a barrel absorbs every convex compact set by [1, Corollary 10.2].

ii) Note that every weak*-compact convex set is strongly bounded.

iii) If E satisfies the condition $[kc]$ and K is totally bounded in E^{*k} then the polar K_o is a k-neighborhood barrel in E and K_o° is compact and convex in E^{*k} by [2, Theorem 2.2]. Hence, $K_o=(K_o^{\circ})_o$ is a neighborhood in E by $[kc]$, and E satisfies the condition $[kt]$.

iv) The similar argument as in iii) can be applied.

v) To show the implication $[bc]\rightarrow[bt]$, let T be a totally bounded set in E^{*b}. Then, T is totally bounded in E^{*t} and T_o° is compact convex in E^{*t} by [2, Theorem 2.2]. So, it is complete in E^{*b} by [3, Theorem 6.1.13], and it is compact convex in E^{*b}. Hence, T_o is a neighborhood by the condition [bc].

PROPOSITION 2.6.

i) *Every metrizable locally convex space satisfies the condition* [kb].

ii) *If* $e: E \to E^{*\tau *\sigma}$ *is surjective and open, then the space* $E^{*\tau}$ *has* [$\sigma\tau$].

iii) *If* E *is a Banach space, then the spaces* E^{*c}, E^{*k} *and* E^{*t} *satisfy the condition* [ct].

iv) *If* E *has the convex compactness property with respect to the weak topology, then the space* E^{*m} *satisfies the condition* [pk], *where* m *denotes the Mackey topology, that is, the uniform topology on the weakly compact convex sets in* E.

Proof. i) Let B be bounded in E^{*k} and $\{x_n\}$ be a sequence in E converging to 0. Then, the set $K=\{x_n\} \cup \{0\}$ is compact, and so, B_o absorbs the sequence $\{x_n\}$ by Lemma 2.3. Hence, B_o is a neighborhood by [1, Theorem 22.1].

ii) If K is a τ-set in $(E^{*\tau})^{*\sigma}$ then $K_o=(e^{-1}(K))^o$ is a neighborhood in $E^{*\tau}$ because $e^{-1}(K)$ is still a τ-set in E.

iii) Note that E satisfies the condition [kc] and so $e: E \to E^{*k*c}$ is a topological isomorphism (cf. [2, Theorem 3.1]). If K is totally bounded in $(E^{*k})^{*c}$ then the closure $\bar{K}$ of K is compact because E is complete. So, $e^{-1}(\bar{K})$ is compact in E, and so $K_o \supseteq (\bar{K})_o=(e^{-1}(\bar{K}))^o$ is a nieghborhood in E^{*k}. This shows that the space E^{*k} satisfies the condition [ct], and the same arguments are applied for E^{*c} and E^{*t}.

iv) Note that the topology of E^{*m*p} is just the weak topology of E in the correspondence $e: E \to E^{*m*}$. If K is compact in $(E^{*m})^{*p}$ then K_o^o is also compact in $(E^{*m})^{*p}$ by the convex compactness property. Hence, $K_o=K_o{}^o{}_o=(e^{-1}(K_o^o))^o$ is a neighborhood in E^{*m} because $e^{-1}(K_o^o)$ is weakly compact convex in E.

3. Examples

EXAMPLE 3.1. An example of metrizable (hence [kb]) space which does not satisfy the condition [ck].

We denote by Φ the space of all finite sequences with the supreme norm. First, we show that if K is a weakly compact convex set containing 0 in Φ then there exists a natural number N such that

$$n>N \text{ and } x\in K \text{ imply } x(n)=0. \tag{3.1}$$

Indeed, assuming that there exists no such N, we have a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers and a sequence $\{x_k\}_{k=1}^{\infty}$ of K such that

i) $x_k(n_k)\neq 0$ for $k=1,2,\ldots$,

ii) $x_k(n_l)=0$ for $l>k$,

iii) $|x_k(n_l)|\leq 1$ for $k,l=1,2,\ldots$.

Then $\{\sum_{k=1}^{n} 2^{-k}x_k : n=1,2,\cdots\}$ is a weak Cauchy sequence which does not converge in K. This contradicts the weak compactness of K.

Now, we show that the space Φ does not satisfy the condition [ck]. Define $f_n\in E^*$ by $f_n(x)=x(n)$ for $n=1,2,\ldots$. Then the set $L=\{0\}\cup nf_n ; n=1,2,\ldots\}$ is compact in E^{*c}. Indeed, if K^o is a basic neighborhood in E^{*c} with convex compact K in E then there exists N with the property (3.1). So, $f_n(x)=0$ for all $x\in K$ and $nf_n\in K^o$, for every n with $n>N$. But, $L_o=\{x\in\Phi ; |x(n)|\leq\frac{1}{n}, n=1,2,\ldots\}$ is not a neighborhood in Φ.

EXAMPLE 3.2. *An example of non-Mackey space which satisfies the condition* [ct].

The space $(l^2)^{*k}$ satisfies the condition [ct] by iii) of Proposition 2.6. We have already shown that $(l^2)^{*k}$ is not a Mackey space in [2, Example 4.1].

EXAMPLE 3.3. An example of non-quasibarrelled space which satisfies the condition [pk].

Let E be a non-reflexive (in the usual sense) Banach space. Then the space E^{*m} has [pk] by iv) of Proposition 2.6 (cf. [3, Theorem 14.2.4]). It is easy to see that E^{*m} is not quaisibarrelled (cf. [1, Exercise 20.A]).

EXAMPLE 3.4. The condition [kk] does not imply the condition [cc].

First, we show that the topologies of the spaces $(\Phi, w)^{*c}$ and $(\Phi, w)^{*p}$ are same, where (Φ, w) is the space Φ with the weak topology. Let K be a compact convex set in (Φ, w) with N as in (3.1). Put $a_i=\sup\{|x(i)|\ ;\ x\in K\}$ and $P=\{Na_1e_1,\ Na_2e_2, \dots, Na_Ne_N\}$, where e_i is an element of Φ given by $e_i(j)=\delta_{ij}$. It suffices to show that $P^o\subseteq K^o$, which implies that K^o is also a neighborhood in $(\Phi, w)^{*p}$. If $f\in P^o$ then $|f(e_i)|\leq\frac{1}{Na_i}$ for each $i=1, 2, \dots, N$. Hence, for each $x\in K$, we have $|f(x)|=|\sum_1^N x(i)f(e_i)|\leq\sum_1^N a_i\frac{1}{Na_i}\leq 1$, and so, $f\in K^o$.

Let (Φ, γ) be denote the locally convex space Φ with the topology generated by k-neighborhood barrells in (Φ, w). Then, (Φ, γ) satisfies the condition [*kk*]. To show that (Φ, γ) does not satisfy the condition [*cc*] let U be the unit ball in Φ. Then the polar U^o is compact in $(\Phi, w)^{*p}$ by the Banach–Alaoglu Theorem, and so, it is also compact in $(\Phi, w)^{*c}$ by the above argument. Note that γ–compactness implies w–compactness, and the topology of $(\Phi, w)^{*c}$ is stronger than the topology of $(\Phi, \gamma)^{*c}$. Hence, U^o is compact convex in $(\Phi, \gamma)^{*c}$. To show that (Φ, γ) does not have [*cc*], it suffices to show that $U^o{}_o=U$ is not a neighborhood in (Φ, γ), that is, U is not a k-neighborhood in (Φ, w). To do this, note that the set $K=\{2e_n\}\cup\{0\}$ is compact in (Φ, w). If U were a k-neighborhood, then there exists a weak–neighborhood V such that $V\cap K\subseteq U\cap K=\{0\}$, and so, $2e_n\notin V$ for any $n=1, 2, \dots$ This is a contradiction because the sequence $\{2e_n\}$ converges weakly to zero.

EXAMPLE 3.5. The condition [*tt*] does not imply the condition [*kk*].

Let K be a compact set in $l^p, 1\leq p<\infty$, and put $\alpha_n=\sup\ \{|x(n)|\ ; x\in K\}$, for each $n=1, 2, \dots$ We show that the sequence $\{\alpha_n\}$ converges to zero. Assuming not, we have a strictly increasing sequence $\{n_k\}$ of natural numbers such that $\alpha_{n_k}>\varepsilon_0$ for each $k=1, 2, \dots$ Choose $x_k\in K$ such that $|x_k(n_k)|\geq\varepsilon_0$. for $k=1, 2, \dots$ Now, we fix k and show that each $\frac{\varepsilon_0}{2}$–ball of x_k contains only finitely many points of $\{x_k\}$. Indeed, there exists a natural number N such that $n\geq N$ implies $|x_k(n)|\leq\frac{\varepsilon_0}{2}$. Hence, if $n_l\geq N$ then we have

$$\|x_l - x_k\|_p \geq |x_l(n_l) - x_k(n_l)| \geq |x_l(n_l)| - |x_k(n_l)| \geq \frac{\varepsilon_0}{2}.$$

So, the sequence $\{x_k\}$ has no convergent subsequence, and we get a contradiction.

Let γ be the locally convex topology on l^1 generated by t-neighborhood barrells in (l^1, w), where w denotes the weak topology. Then, (l^1, γ) satisfies the condition $[tt]$. Put

$$U = \{x \in l^1 \; ; \; |x(n)| \leq 1, \; n = 1, 2, \ldots\}.$$

First, we show that U is a k-neighborhood barrel. If K is γ-compact then it is w-compact, and so, K is norm-compact by [3, Example 9. 5. 5.]. So, by the argument in the above paragraph, there exists a natural number N such that $|x(n)| \leq 1$ whenever $n \geq N$ and $x \in K$. Put $V = \{x \in l^1 \; ; \; |x(n)| \leq 1, \; n = 1, 2, \ldots, N\}$. Then, V is a γ-neighborhood and we have $V \cap K \subseteq U \cap K$. It is clear that U is a barrel.

To show that U is not a neighborhood in (l^1, γ), it suffices to show that U is not a t-neighborhood in (l^1, w). Note that boundedness and total boundedness coincides in the weak topology (cf. [3, Exercise 8. 2. 6.]). So, the set $B = \{x \in l^1 \; ; \; \|x\|_1 \leq 2\}$ is w-totally bounded. If U were a t-neighborhood in (l^1, w), then there exists a finite set F in l^∞ such that $F_o \cap B \subseteq U \cap B$, which implies $F_o^\circ \cup B^\circ \supseteq U^\circ \cup B^\circ$ in l^∞. Now, each element $e_n \in l^\infty$ is contained in $U^\circ \backslash B^\circ$ for $n = 1, 2, \ldots$, but F_o° is finite-dimensional, and we have a contradiction. Hence, U is not a γ-neighborhood, and the space (l^1, γ) does not satisfy the condition $[kk]$.

EXAMPLE 3. 6. The condition $[tb]$ does not imply the condition $[kb]$.

Note that the polar U° of the set

$$U = \{x \in l^1 \; ; \; |x(n)| \leq \frac{1}{n}, \; n = 1, 2, \ldots\}$$

is k-absorbing in (l^1, w). Indeed, if K is compact in (l^1, w) then it is norm-compact, and so, U absorbs $K^\circ{}_o$ by [1, Corollary 10. 2]. By the similar argument as in Examples 3. 4 and 3. 5, it suffices to show that U is not t-absorbing. Indeed, the unit ball B is totally bounded in (l^1, w), but it is easy to see that U does not absorb B by considering the sequence $\{e_1, e_2, \ldots\}$ in B.

EXAMPLE 3. 7. An example of a quaisibarrelled space which does not satisfy the condition [tb].

It is easy to see that if U and V are t-absorbing barrels in E and F, respectively, then $U\times V$ is also a t-absorbing barrel in $E\oplus F$. Note that the unit ball U in c_0 is a t-absorbing barrel in (c_0, w) because a totally bunded set in (c_0, w) is norm-bounded. Also, the set

$$V=\{x\in c_0\ ;\ |x(n)|\leq\frac{1}{n},\ \ n=1,2,\ldots\}$$

is t-absorbing in (c_0, n). Indeed, if T is a toally bounded set in (c_0, n) then $T^{\circ}{}_{o}$ is compact convex, and so V absorbs $T^{\circ}{}_{o}$ by [1, Corollary 10.2]. Hence, $U\times V$ is a t-absorbing barrel in $(c_0, w)\oplus(c_0, n)$. Now, because V is not b-absorbing in (c_0, n), the set $U\times V$ is not b-absorbing in $(c_0, w)\oplus(c_0, n)$, we can apply the similar argument as in Examples 3.4 and 3.5.

Example 3.8. An example of locally convex space which does not satisfy the condition [bc].

Note that $e: l^2\rightarrow(l^2)^{*p*b}$ is a topological isomorphism. Put $K=\{0, e_1, \ldots, \frac{1}{n}e_n, \ldots\}$, where $\{e_n\}$ is an orthonormal basis. Then, the convex extension $\hat{K}$ of K is compact, and so, $e(\hat{K})$ is compact convex in $((l^2)^{*p})^{*b}$. But, the polar $e(\hat{K})_o$, a subset of K°, is not a neighborhood in $(l^2)^{*p}$. Hence, $(l^2)^{*p}$ does not satisfy the condition [bc].

Remark. By the above examples, we cannot put any more arrow in Figure 2, except possible implications [pk]→[ct], [ck]→[ct], [bc]→[tt] and k-reflexive→[ct]. In order to deal with these implications, we need to find an intrinsic characterization of the polars of compact or totally bounded sets in E^{*c} as in the cases of $\tau=k, t$, which was done in [2, Theorem 2.2].

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