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A NECESSARY CONDITION FOR THE LOCAL SOLVABILITY OF THE SYSTEM OF THE LEWY TYPE VECTOR FIELDS

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0. Introduction

Treves [8], dealed with the local solvability of the overdetermined system of the following Mizohata type vector fields in an open subset of R^{m+1}

$$K_j=\frac{\partial}{\partial t_j}+k_j(t,u)\frac{\partial}{\partial u},\quad j=1,\cdots,m, \tag{0.1}$$

with analytic coefficients and satisfying Frobenius condition, and shows that the necessary and sufficient condition for the local solvability is Condition (P). (cf. Treves [8])

But for the local solvability of the system of the Lewy type vector fields defined in an open subset contained in an analytic manifold Ω of dimension $2m+1$, of the type

$$L_j=\frac{\partial}{\partial \bar{z}_j}+\lambda_j(t,y,u)\frac{\partial}{\partial u},\quad j=1,\cdots,m(m\geq 2),\quad z_j=t_j+iy_j, \tag{0.2}$$

$t=(t_1,\cdots,t_m)\in R^m$, $y=(y_1,\cdots,y_m)\in R^m$, $u\in R^1$, with analytic coefficients, the sufficient condition is known only when Ω is the hypersurface of C^{m+1}, of the form $\Omega=\{(z_1,\cdots,z_m,w);\ w=u+i\phi(z,\bar{z},u), \phi$ is real valued, analytic function$\}$. This sufficient condition is $Y(1)$ condition which measures the convexity of the domain Ω in terms of the Levi form (cf. Airapetyan and Khenkin [1], and Folland and Kohn [3]).

This condition, however, is quite abstract and needs a concrete interpretation. Moreover, even if this sufficient condition is satisfied, the concrete integral representation of the local solution of the equation

$$L_ju=f_j,\quad j=1,\cdots,m\quad (m\geq 2), \tag{0.3}$$

is unknown.

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In this paper our goal is to show that condition $(\tilde{P})$ is necessary for the local solvability of the system of the Lewy type vector fields (0.2).

1. Basic concepts and main results

Throughout this paper Ω denotes an analytic manifold of dimension $2m+1$, countable at infinity. Here analytic means real analytic, and complex analytic means holomorphic. An abstract analytic *CR* structure of codimension one is the datum of an analytic vector subbundle T of the complex tangent bundle $CT\Omega$ submitted to the following three conditions:

(1.1) $[T, T] \subset T$, i.e., the commutation bracket of any two analytic sections of T over an open subset of Ω is a section of T over that same subset;

(1.2) $T \cap \bar{T} = \{0\}$ ($\bar{T}$ is the complex conjugate of T);

(1.3) the fiber dimension of T over C is equal to m.

An abstract *CR* structure is the structure which is obtained if we replace analytic by C^∞ in the above definition. Let T be the orthogonal of T in the complex cotangent bundle $CT^*\Omega$ for the duality between tangent and cotangent vectors. Then the fiber dimension of T is equal to $m+1$. Note that (1.2) is equal to

$$CT^*\Omega = T' + \bar{T}'. \tag{1.4}$$

Let Ω' be any open subset of Ω. A C^1 function (resp. a distribution) f in Ω' is called an analytic *CR* function (resp. an analytic *CR* distribution) if $Lf = 0$ whatever the analytic section L of T over Ω'. The differentials of the analytic *CR* functions are sections of T'. The abstract analytic *CR* structure is locally integrable if at any point p of Ω there are $m+1$ germs of analytic *CR* functions whose differentials at p are linearly independent (and thus make up a linear basis of T'_p) (for the definitions see [5] and [7]).

Now, assume that we are given an abstract analytic *CR* structure T on an analytic manifold Ω of dimension $2m+1$ ($m \geq 2$). Let U be an open neighborhood of an arbitrary point p of Ω in which there are (real) local coordinates $t_1, \cdots, t_m, y_1, \cdots, y_m, u$ and $m+1$ analytic CR functions $z_1, \cdots, z_m$, w such that

$$(1,5) \qquad z_j = t_j + iy_j \ (i=\sqrt{-1},\ j=1, \cdots, m, m\geq 2);$$

$$(1.6) \qquad w = u + i\phi(t, y, u),$$

$$\phi \text{ real value, } \phi(0,0,0)=0,\ d_u\phi(0,0,0)=0,$$

and of course, ϕ analytic in U. Actually we may even assume

$$(1.7) \qquad \phi(0,0,u)\equiv 0.$$

We shall always assume that the coordinates and CR functions (1.5), (1.6) all vanish at the point p. Henceforth we refer to it as the origin. It is convenient to assume that

$$(1.8) \qquad U = B_r \times J,$$

where B_r is the open ball $\{(t, y)\in R^{2m};\ \sqrt{|t|^2+|y|^2}<r\}$, and J an open interval in the real line containing the origin. We shall also assume that the closure of U in Ω, Cl U, is compact.

We shall denote by Z the mapping

$$(t, y, u) \mapsto Z(t, y, u) = (z_1, \cdots, z_m, w(t, y, u))$$

from U to C^{m+1}. The image $w(U)$ is the union of a collection of intevals $\{u_0\}\times I(u_0)$, $u_0\in J$ where $I(u_0)$ is the image of B_r via the map $(t, y)\mapsto \phi(t, y, u_0)$. Of course $I(u_0)$ is always an inteval containing zero, but otherwise fairly arbitrary. In particular it is reduced to zero whenever $\phi(t, y, u_0)\equiv 0$.

Note that dz_j $(j=1, \cdots, m)$, dw make up a linear basis of T'_p at every point p of U, and that $dz_j, d\bar{z}_k$ $(j, k=1, \cdots, m)$, dw make up a linear basis of $CT_p^*\Omega$ at every point p of U. In U the abstract analytic CR structure T is generated by m analytic vector fields L_j, $j=1, \cdots, m$ $(m\geq 2)$, such that

$$(1.9) \qquad L_j Z_k = L_j w = 0,\ j, k=1, \cdots, m.$$

If we further require

(1.10) $L_j\bar{z}_k=\delta_{jk}$ (Kronecker's index), $j, k=1, \cdots, m$,

the L_j are uniquely determined, since $dz_1, \cdots, dz_m, d\bar{z}_1, \cdots, d\bar{z}_m, dw$ span the whole cotangent space $CT_p^*\Omega$ at every point p of U. We have

(1.11) $$L_j=\frac{\partial}{\partial\bar{z}_j}+\lambda_j(t, y, u)\frac{\partial}{\partial u}, \quad j=1, \cdots, m, \quad z_j=t_j+iy_j.$$

Of course,

$$\lambda_j=-i\phi\bar{z}_j/(1+i\phi_u),$$

where subscripts mean differentiation. Note that, by (1.9), we have $L_j\bar{w}=L_j(w+\bar{w})=L_j(2u)=2\lambda_j$, i.e.,

(1.12) $$\lambda_j=\frac{1}{2}L_j\bar{w}.$$

Introducing the vector fields

(1.13) $L_0=w_u^{-1}\frac{\partial}{\partial u}$ and $M_j=\frac{\partial}{\partial z_j}+\mu_j\frac{\partial}{\partial u}$, $j=1, \cdots, m$,

where $$\mu_j=-i\phi z_j/(1+i\phi_u),$$

we have

(1.14) $M_j\bar{z}_k=M_jw=0$, $M_jz_k=\delta_{jk}$, if $j, k=1, \cdots, m$,

(1.15) $$L_0z_k=0, \quad k=1, \cdots, m,$$
$$L_0w=1.$$

Thus $L_1, \cdots, L_m, L_0, M_1, \cdots, M_m$ is the basis in $CT_p\Omega$ $(p\in U)$, dual of the basis $dz_1, \cdots, dz_m, d\bar{z}_1, \cdots, d\bar{z}_m, dw$ of $CT_p^*\Omega$.
From (1.9)-(1.10), (1.14)-(1.15) we have, in U,

(1.16) $$[L_j, L_k]=[L_j, M_l]=[M_l, M_m]=0,$$
$$j, k=0, 1, \cdots, m, \quad l, m=1, \cdots, m.$$

These commutation relations are equivalent to the equations

(1.17) $$L_j\lambda_k=L_k\lambda_j, \quad L_j\mu_k=M_k\lambda_j, \quad M_j\mu_k=M_k\mu_j,$$
$$\text{if } j, k=1, \cdots, m,$$

(1.18) $L_j w_u^{-1} = L_0 \lambda_j, \quad M_j w_u^{-1} = L_0 \mu_j, \quad j=1, \cdots, m.$

If F is a C^1 function in U, we have

$$dF = \sum_{j=1}^{m} L_j F d\bar{z}_j + \sum_{k=1}^{m} M_k F dz_k + L_0 F dw.$$

We shall need the results from [7] concerning the solutions of the homogeneous equations

(1.19) $L_j h = 0, \quad j=1, \cdots, m.$

We restate the main theorems of [7]. Set $U' = B_{r'} \times J'$, with $0 < r' < r$, and J' an open interval whose compact closure is contained in J.

THEOREM I. *Let h be a continuous solution of* (1.19) *in some open neighborhood of $Cl\ U'$. Then h is the uniform limit, in $Cl\ U'$, of a sequence of polynomials, with complex coefficients, in $Z(t, y, u)$.*

THEOREM II. *Let h be a distribution solution of* (1.19) *in some open neighborhood of $Cl\ U'$. There are, then, an integer $q \geq 0$ and a C^1 solution of* (1.19) *in a neighborhood of $Cl\ U'$, f, such that*

$$h = \left(\sum_{i=1}^{m} M_i^2 + L_0^2\right)^q f.$$

Combining Theorems I and II we see that any distribution solution of (1.19) is the limit, in the distribution sense, in U, of a sequence of polynomials in $Z(t, y, z)$.

Note that h is constant on the fibers of the map Z in U', that is, on the set

$$\{(t, y, u) \in U' \ ; \ Z(t, y, u) = z_0\},$$

for any given point z_0 in C^{m+1}. Because of the peculiar from of the map Z (see (1.5), (1.6)), we need only consider the fiber of w, which is given by

$$\{(t, y, u) \in U' \ ; \ u = u_0, \ \phi(t, y, u_0) = v_0\}, \quad u_0 + iv_0 \in C,$$

and which can thus be identified to a subset of the ball B_r.

DEFINITION 1.1. We shall say that the system $L = (L_1, \cdots, L_m)$ satisfies Condition $(\tilde{P})$ at a point p of U if there is a basis of neigborhoods of p in U, in each one of which the fibers of w are connected.

We shall say that L satisfies Condition $(\tilde{P})$ in U if it satisfies Condition $(\tilde{P})$ at every point of U.

We shall be concerned with the inhomogeneous equations

$$(1.20)\qquad L_j h = f_j,\quad j=1,\cdots,m,$$

where $f_1,\cdots,f_m$ are C^∞ functions near p_0 satisfying the compatibility conditions:

$$(1.21)\qquad L_j f_k = L_k f_j,\quad j,k=1,\cdots,m.$$

We have the following necessary condition for the local solvability of the system of the Lewy type vector fields (1.20).

THEOREM. *Suppose that the system* $L=(L_1,\cdots,L_m)$ $(m\geq 2)$ *does not satisfy Condition* $(\tilde{P})$ *at the point* p_0 *of* U *and that every distribution solution of the homogeneous equations* $L_j u=0$ *in* U, $j=1,\cdots,m$, *is a continuous function.*

Then there is a C^∞ function f in an open neighborhood $V\subset U$ of p_0, vanishing to infinite order at p_0 such that

(1.22) the functions $f_j=\lambda_j f$, $j=1,\cdots,m$ (see (1.11)) satisfy the compatibility conditions (1.21) in V. Furthermore, given any open neighborhood $W\subset V$ of p_0,

(1.23) no distribution h in W satisfies (1.20).

The proof of Theorem will be given in Section 2.

We reformulate Condition $(\tilde{P})$ (cf. Definition 1.1) in the following manner:

(1.24) Every open neighborhood $V_p\subset U$ of p contains another open neighborhood W_p of p which intersects at most one connected component of every fiber of w in V_p.

Indeed, suppose first that V_p contains a neighborhood W_p' of p in which every fiber of w is connected. Then we can take W_p in (1.24) to be the interior of W_p'. Conversely, suppose that (1.24) holds; call W_p' the union of all the connected components of the fibers of w in V_p which intersect W_p.

We shall reason in (t, y)-space R^{2m} for a fixed u. We denote by B, B', B'' three open balls centered at the origin in R^{2m}, such that

$$B'' \subset B' \subsetneqq B. \tag{1.25}$$

We shall look at a real valued analytic function φ in B. If A is any subset of B and c any real number we write

$$\begin{aligned} A^+(c) &= \{(t, y) \in A\ ;\ \varphi(t, y) > c\}, \\ A^-(c) &= \{(t, y) \in A\ ;\ \varphi(t, y) < c\}, \\ A^0(c) &= \{(t, y) \in A\ ;\ \varphi(t, y) = c\}. \end{aligned} \tag{1.26}$$

In other words A^0, A^+, A^- are the level, superlevel and sublevel sets, respectively, of the function φ in A.

We reintroduce the variable u. If A is any subset of $U = B_r \times J$ and u, v any pair of real numbers, we write

$$\begin{aligned} A^+(u, v) &= \{p \in A\ ;\ u(p) = u, \phi(p) > v\}, \\ A^-(u, v) &= \{p \in A\ ;\ u(p) = u, \phi(p) < v\}. \end{aligned} \tag{1.27}$$

Here $A^+(u, v)$ or $A^-(u, v)$ might be empty, for $u \notin J$.

Proposition 1.1. *Property* (1.24) *is equivalent to each one of the following properties*:

(1.28) *Every open neighborhood $V_p \subset U$ of p contains another open nerghborhood of p, W_p, such that, given any pair of real numbers u, v, W_p intersects at most one connected component of $V_p^+(u, v)$, and at most one of $V_p^-(u, v)$.*

(1.29) *Every open neighborhood $V_p \subset U$ of p contains another open neighborhood of p, W_p, such that any two points in W_p, of the kind (t_0, y_0, u), (t_1, y_1, u) can be joined by a piecewise analytic curve in V_p on which u is constant and ϕ monotone.*

For the proof see [6].

2. The proof of Theorem

Our starting point will be the hypothesis that Condition $(\tilde{P})$ is not satisfied at the origin (cf. Definition 1.1). Actually it is convenient to make use of the version (1.28) of $(\tilde{P})$, or rather of its negation.

Let us for instance assume that the following property holds:

(2.1) There is an open neighborhood $V \subset U$ of the origin, and a sequence of points in C, $z_{m+1_\nu} = u_\nu + iv_\nu$, $\nu = 1, 2, \cdots$, converging to zero, such that any neighborhood of the origin, $W \subset V$, intersects two distinct connected components of $V^+(u_\nu, v_\nu)$ (see (1.27)) for some ν.

Note that (2.1) remains valid if we decrease V. Thus we shall assume that $V \subset\subset U = B_r \times J$, and that $V = B_{r_0} \times J_0$. Possibly after a change of subscripts $\nu = 1, 2, \cdots$, we select a sequence of open neighborhoods

$$W_\nu = B_{r_\nu} \times J_\nu, \tag{2.2}$$

with $r_0 > r_\nu \searrow +0, J_\nu =]-r_\nu, r_\nu[$, such that, for each ν, W_ν intersects at laest two distinct connected components of $V^+(u_\nu, v_\nu)$, $C_{1\nu}$ and $C_{2\nu}$.

Fix u_0 in J. Then the number of critical values of the mapping $w(t, y, u)$ in Cl V that lie on the vertical line $\mathrm{Re}\, z_{m+1} = u_0$ is finite. Indeed, they are the values of w on the set of points (t, y, u) in Cl V such that

$$u = u_0, \quad d_{(t, y)}\phi(t, y, u_0) = 0. \tag{2.3}$$

But in the neighborhood of Cl V the equations (2.3) define an analytic set, of which only finitely many connected components intersect the compact set Cl V, and w is constant on each of these components. This implies that, for each ν, there is $v'_\nu > v_\nu$ such that the fiber of w in Cl V,

$$F(z'_{m+1_\nu}) = \{(t, y, u) \in \mathrm{Cl}\ V\ ;\ w(t, y, u) = z'_{m+1_\nu} = u_\nu + iv'_\nu\} \tag{2.4}$$

intersects both $W_\nu \cap C_{1\nu}$, and $W_\nu \cap C_{2\nu}$ and such that z'_{m+1_ν} is not a critical value of w in Cl V. But then W_ν must intersect two distinct components of $F(z'_{m+1_\nu})$. In other words, we may start with the following hypothesis:

(2.5) There is a totally ordered basis of open neighborhoods of the origin, $W_\nu \subset W$, and a sequence of complex numbers z_{m+1_ν}, converging to zero, none of which is a critical value of $w(t, y, u)$ in Cl V, such that, for each ν, W_ν intersects two distinct connected components of the fiber $F(z_{m+1_\nu})$.

For each $\nu=1, 2, \cdots$, we select a closed disk D_ν, centered at $z_{m+1\nu}$, with radius $d_\nu>0$. In the argument below we shall decrease d_ν a finite number of times. First of all we select d_ν small enough that the following conditions are fulfilled:

(2.6) For each ν, D_ν is entirely contained in the (open) set of noncritical values of $w(t, y, u)$ in Cl V, and in the interior of the image $w(W_\nu)$;

(2.7) the projections of the D_ν into the real axis are pairwise disjoint.

For each ν, let C_ν^+ and C_ν^- denote two distinct compontents of $F(z_{m+1\nu})$ which intersect W_ν. Possibly after decreasing d_ν we may make the following assumption:

(2.8) There are two analytic submanifolds of dimension two, Σ_ν^+ and Σ_ν^-, which intersect respectively C_ν^+ and C_ν^-, and whose closures are disjoint compact subsets of W_ν, each mapped diffeomorphically onto D_ν by w.

And possibly after some more decreasing of d_ν we select two open neighborhoods of C_ν^+ and C_ν^- respectively, in U, β_ν^+ and β_ν^-, endowed with the following properties:

(2.9) $$(\mathrm{Cl}\ \beta_\nu^+) \cap (\mathrm{Cl}\ \beta_\nu^-)=\phi;$$

(2.10) $\Sigma_\nu^+\subset\beta_\nu^+$, $\Sigma_\nu^-\subset\beta_\nu^-$ and the image via w of β_ν^+, as well as that of β_ν, is exactly equal to Int D_ν;

(2.11) any connected component of a fiber $F(z_{m+1})$ of w in Cl V which intersects $\beta_\nu^\pm$ is entirely contained in $\beta_\nu^\pm$;

(2.12) no two distinct connected components of the same fiber $F(z_{m+1})$ intersects either β_ν^+ or β_ν^-.

For each $\nu=1, 2, \cdots$, let r'_ν be a number such that $r_\nu<r'_\nu<r_{\nu-1}$ and set $W'_\nu=B_{r'_\nu}\times J'_\nu$, $J'_\nu=[-r'_\nu, r'_\nu]$. We cosider a distribution h in W'_ν which is a solution of the inhomogeneous equations (1.20). We shall assume that the righthand sides are continuous functions in V, and satisfy (1.21) in V. Furthermore we assume that

(2.13) $$\operatorname{supp} f_j \subset w^{-1}(0) \cup \bigcup_{\nu=1}^{+\infty} \beta_\nu^+.$$

We can easily check that the set at the right has an intersection with Cl V that is closed. Note also that we have

(2.14) $$L_j h=0, \quad j=1, \cdots, m,$$

in the set

(2.15) $$W_\nu' \backslash \mathrm{Cl}\Big(\bigcup_{\nu=1}^{+\infty} \beta_\nu^+\Big).$$

We introduce, for each $\nu=1, 2, \cdots$, a closed disk D_ν', also centered at $z_{m+1\nu}$, with radius $d_\nu'>d_\nu$, such that the properties analogous to (2.6), (2.7), (2.8) hold. We call A_ν the annulus $D_\nu' \backslash D_\nu$.

Note that, by the assumption of Theorem, h is a continuous function in the set

$$U_\nu = W_\nu \cap w^{-1}(A_\nu).$$

The key to the proof of Theorem 1.1 lies in the following assertion:

(2.16) h is constant on the fibers of $Z=(z_1, \cdots, z_m, w)$ in U_ν.

Proof of (2.16): We note that (2.14) holds in the set

(2.17) $$\{(t, y, u) ; \sqrt{|t|^2+|y|^2}<r_\nu, \ d_\nu<|u-u_\nu|<d_\nu'\}.$$

We apply Theorems I, II taking $U'=B_{r'} \times J'$ to have compact closure contained in (2.17). According to the assumption of Theorem h is a continuous function. Therefore we conclude that h is the C^1 limit of a sequence of polynomials of $Z(t, y, u)$ in the intersection of (2.17) with U_ν. Thus h must be constant on the fibers of $Z(t, y, u)$ in that intersection.

Let us call $\mathcal{D}$ the interior of the subset C of $w(U_\nu)$ such that

(2.18) h is constant on the fibers of Z in $w^{-1}(C) \cap U_\nu$.

We have just shown that $\mathcal{D}$ contains the set

(2.19) $$z_{m+1} \in \mathrm{Int}\, A_\nu, \ |\mathrm{Re}\, z_{m+1}-u_\nu|>d_\nu, \quad z_{m+1}=u+iv \in C$$

Suppose now that there is a point z^*_{m+1} in the boundary of $\mathcal{D}$ with respect to A_ν. We apply once again Theorem I, availing ourselves of the fact that h is a solution of (2.14) in an open neighborhood of $S^*=F(z^*_{m+1})\cap \mathrm{Cl}\, W_\nu$. There is a number $\delta>0$ such that every point $P^*\in S^*$ is the center of an open ball with radius δ in which h is the C^1 limit of a sequence of polynomials in $Z(t, y, u)$. Note that the sequence in question may change from point to point. We may suppose that the union of all those balls is contained in a compact subset K of $W'_\nu(\supset W_\nu)$. The restriction of w to K is open, and therefore there is a clsed disk D^* centered at z^*_{m+1} with the following property:

$$(2.20)\qquad U_\nu\cap w^{-1}(D^*)\subset\{p\in W_\nu\ ;\ \mathrm{dist}\,(p, S^*)\leq\delta\}.$$

Let then $P_j\in U_\nu$ be such that $w(p_j)=\zeta\in D^*$ $(j=1, 2)$. We can find $p^*_j\in S^*$ such that $|p^*_j-p_j|\leq\delta$, and there is a continuous function $\tilde{h}_j$ in $\{(z_1, z_2, \cdots, z_m, w)\ ;\ \ w=u+i\phi(t, y, u)\in D^*,\ \ (z_1, \cdots, z_m, u)=(t, y, u)\in U_\nu \cap w^{-1}(D^*)\}$ such that $h=\tilde{h}_j\circ Z$ in the ball centered at p^*_j with radius δ. Moreover, $\tilde{h}_j$ is holomorphic in $\{(z_1, \cdots, z_m, w)\ ;w=u+i\phi(t, y, u)\in D'^*,\ \ (z_1, \cdots, z_m, u)\in U_\nu\cap w^{-1}(D'^*)\}$, where D'^* is an open disk contained in D^*, also centered at z^*_{m+1}, and can be selected independently of the point p^*_j on S^*. Let $(\tilde{h}_j)_{z_{m+1}}$ the restrictions of $\tilde{h}_j$ to z_{m+1}-plane, where $z_{m+1}=u+iv$. Note that $(\tilde{h}_j)_{z_{m+1}}$ is holomophic in D'^*. But since $(\tilde{h}_1)_{z_{m+1}}=(\tilde{h}_2)_{z_{m+1}}$ in $D'^*\cap\mathcal{D}$, we must have $(\tilde{h}_1)_{z_{m+1}}=(\tilde{h}_2)_{z_{m+1}}$ in D'^*, and therefore $D'^*\subset\mathcal{D}$, which contradicts the fact that its center is a boundary point of $\mathcal{D}$. We must therefore have $\mathcal{D}=A_\nu$.

We draw right away a consequence of (2.16). Because of the validity of (2.8) when D'_ν is substituted for D_ν, We see that

$$w(U_\nu)=A_\nu \text{ and}$$
$$Z(U_\nu)=\{(z_1, \cdots, z_m, w)\ ;w=u+i\phi(t, y, u)\in A_\nu,$$
$$(z_1, \cdots, z_m, u)=(t, y, u)\in U_\nu\}.$$

Therefore there is a continuous function in $Z(U_\nu)$, $\tilde{h}$, holomophic in the interior of $Z(U_\nu)$, such that $h=\tilde{h}\circ Z$ in U_ν. Let $(\tilde{h})_{z_{m+1}}$ be the restriction of $\tilde{h}$ to the z_{m+1}-plane, $z_{m+1}=u+iv$. Then $(\tilde{h})_{z_{m+1}}$ is a continuous function in A_ν and holomorphic in the interior of A_ν. We contend that

(2.21) $(h)_{z_{m+1}}$ extends holomorphically to the interior of D'_ν.

Indeed call Σ'^-_ν the analogue of Σ^-_ν (see (2.8)) when D'_ν is substituted for D_ν. Since h is a continuous solution of the system of equtions (2.14) in some open neighborhood of $\Sigma_\nu'^-$, its restriction to $\Sigma_\nu'^-$ is continuous. Let $\tilde{\tilde{h}}$ be the push forward of the restriction of h to $\Sigma_\nu'^-$ via Z; it defines a real analytic function in the interior of $\{(z_1, \cdots, z_m, w); w=u+i\phi(t, y, u) \in D_\nu', \ (z_1, \cdots, z_m, u) \in W_\nu \cap \beta_\nu'^-\}$.

In some open neighborhood of each point of $\Sigma_\nu'^- h$ is a uniform limit of polynomials with respect to Z, by Theorem I, as a consequence of which we see that $\tilde{\tilde{h}}$ must be holomorphic in the interior of $\{(z_1, \cdots, z_m, w) : w=u+\phi(z, \bar{z}, u) \in D_\nu', (z_1, \cdots, z_m, u) \in W_\nu \cap \beta_\nu'^-\}$. Then $(\tilde{\tilde{h}})_{z_{m+1}}$ must be holomophic in the interior of D_ν'. Since $(\tilde{\tilde{h}})_{z_{m+1}} = (h)_{z_{m+1}}$ in A_ν, this proves our assertion.

We can now proceed with the construction of the function f in Theorem.

For each ν, we select an arbitrary closed disk D_ν^* centered at z_{m+1} with radius $d_\nu^* < d_\nu$. Let then $\tilde{f}$ be a function holomorphic with respect to the variables $z_1, \cdots, z_m$, and C^∞ with respect to the variable z_{m+1}, vanishing identically in the complement of

$$\{(z_1, \cdots, z_m, w) \ ; \ w=u+i\phi(t, y, u) \in \bigcup_{\nu=1}^{+\infty} D_\nu^*,$$
$$(z_1, \cdots, z_m, u) = (t, y, u) \in W_\nu \cap w^{-1}(\bigcup_{\nu=1}^{+\infty} D_\nu^*)\},$$

and such, moreover, that

(2.22) for every $\nu=1, 2, \cdots, \tilde{f}>0$ in the interior of

$$\{(z_1, \cdots, z_m, w) \ ; \ w=u+i\phi(t, y, u) \in D_\nu^*,$$
$$(z_1, \cdots, z_m, u) = (t, y, u) \in W_\nu \cap w^{-1}(d_\nu^*)\}.$$

Note that $(\tilde{f})_{z_{m+1}} > 0$ in $\operatorname{Int} D_\nu^*$ for every $\nu=1, 2, \cdots$. Then we define

(2.23) $f=\tilde{f} \circ Z$ in $V \cap (\bigcup_{\nu=1}^{+\infty} \beta_\nu^+)$, $f \equiv 0$ everywhere else.

Cleary f is a function anaytic with respect to the variables $t_1, y_1, \cdots, t_m, y_m$, and C^∞ with respect to the variable u in $V \backslash w^{-1}(0)$, and vanishes to infinite order on $V \cap w^{-1}(0)$; thus $f \in C^\infty(V)$.

We contend that the assertion (1.22) is correct; it suffices to check (1.21) in some neighborhood of an arbitrary point of $V \cap (\bigcup_\nu \beta_\nu^+)$. There $f=\tilde{f} \circ Z$ and therefore

$$L_j f = \frac{\partial f}{\partial \bar{z}_j} \circ Z + \left(\frac{\partial f}{\partial \bar{z}_{m+1}} \circ Z \right) L_j \bar{w} = \left(\frac{\partial f}{\partial \bar{z}_{m+1}} \circ Z \right) 2\lambda_j, \quad j=1, \cdots, m,$$

since $L_j z_k = \delta_{jk}$ and $L_j \bar{w} = 2\lambda_j$. Therefore, if we set $\tilde{f}_1 = 2(\partial \tilde{f} / \partial \bar{z}_{m+1})$, $f_1 = \tilde{f}_1 \circ Z$, we have:

$$L_j f = \lambda_j f_1. \tag{2.24}$$

On the other hand, the commutation relations (1.16) are quivalent to

$$L_j \lambda_k = L_k \lambda_j, \quad j, k=1, \cdots, m. \tag{2.25}$$

Combining (2.24) and (2.25) with the equatios

$$L_j(\lambda_k f) = f L_j \lambda_k + \lambda_k \lambda_j f_1,$$
$$L_k(\lambda_j f) = f L_k \lambda_j + \lambda_j \lambda_k f_1, \quad j, k=1, \cdots, m,$$

yields at once

$$L_j(\lambda_k f) = L_k(\lambda_j f), \quad j, k=1, \cdots, m. \tag{2.26}$$

Next we prove Assertion (1.23).

We shall prove that, given an arbitrary integer $\nu \geq 1$, there is no distribution h satisfying (1.20) in W'_ν.

Observing that the function $\tilde{f}$ used to define f has compact support, set

$$\tilde{v} = \tilde{f} *_{m+1} (1/2\pi z_{m+1}),$$

where $*_{m+1}$ is the convolution of distributions in z_{m+1}-plane. Note that $\tilde{v}$ is a function holomorphic with respect to the vriables $z_1, \cdots, z_m$, and C^∞ with respect to the variable z_{m+1} We have

$$\frac{\partial v}{\partial \bar{z}_{m+1}} = \tilde{f}/2. \tag{2.27}$$

Set $v = \tilde{v} \circ Z$ in V. We have, in a neighborhood u of $(\mathrm{Cl}\, \beta_\nu^+) \cap V$,

$$L_j v = \frac{\partial v}{\partial \bar{z}_j} \circ Z + \left(\frac{\partial v}{\partial \bar{z}_{m+1}} \circ Z \right) L_j \bar{w} = \lambda_j f = f_j, \quad j=1, \cdots, m,$$

by (1.12), and therefore, by (1.20), we have, in $U \cap W'_\nu$,

$$L_j(h - v) = 0, \quad j=1, \cdots, m. \tag{2.28}$$

We have the right to take U such that it contains the surface $\sum_\nu'^+$ analogous to $\sum_\nu^+$ in (2.8), when D'_ν is substituted for D_ν. Once again by the assumption of theorem we know that $h-v$ is a continuous function in some neighborhood of $\sum_{\nu}^{\prime\prime +}$, and its rescriction to $\sum_{\nu}^{\prime\prime +}$ can be pushed forward via Z as a real analytic function $\tilde{w}$ in

$$\{(z_1, \cdots, z_m, w) \in Z(V)\ ;\ w=u+i\phi(t, y, u) \in \text{Int } D_{\nu}^{\prime},\\ (z_1, \cdots, z_m, u) = (t, y, u) \in W_{\nu} \cap \beta_{\nu}^{\prime\prime +}\}.$$

And again, by Theorm I, we know that the latter is a uniform limit of polynornials of $(z_1, \cdots, z_m, z_{m+1})$ in the neighborhood of each point of the set

$$\{(z_1, \cdots, z_m, w) \in Z(V)\ ;\ w=u+i\phi(t, y, u) \in \text{Int } D_{\nu}^{\prime},\\ (z_1, \cdots, z_m, u) = (t, y, u) \in W_{\nu} \cap \beta_{\nu}^{\prime\prime +}\},$$

therefore $\tilde{w}$ is holomorphic in that set. Therefore $(\tilde{w})_{z_{m+1}}$ is holomorphic in the neighborhood of each point of $\text{Int } D_{\nu}^{\prime}$. Since $(h)_{z_{m+1}}$ can be extended holomorphically to $\text{Int } D_{\nu}^{\prime}$, the same must be true of $(\tilde{v})_{z_{m+1}}$. This demands

$$\int_{\partial D_{\nu}} (\tilde{v})_{z_{m+1}} dz_{m+1} = 0, \quad z_{m+1} = u + iv,$$

and therefore, by Stokes' theorem

$$\int_{D_{\nu}} (\tilde{f})_{z_{m+1}} dz_{m+1} \wedge d\bar{z}_{m+1} = 0,$$

which contradicts (2.22).

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