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GLOBAL HOLOMORPHIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A PARAMETER IN A STEIN MANIFOLDS*

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1. Introduction

Let S be a pure finite dimensional Stein space, C^n be the space of n complex variables $z_1, z_2, \dots, z_n$, Ω be a Stein domain of the product space $C^n \times S$ of C^n and S. Let $O_{z,s}$ be the sheaf over Ω of germs of holomorphic functions of variables $z=(z_1, z_2, \dots, z_n) \in C^n$ and $s \in S$, m be a positive integer, $a^i_{pq}(z, s)$ be holomorphic functions on Ω for $i=1, 2, \dots, n$ and $p, q=1, 2, \dots, m$. We define sheaf homomorphisms T_i, P_j and P of $O^m := O^m_{z,s}$ in O^m for $i, j=1, 2, \cdots, n$ by putting

$$T_i f = {}^t\Big(\frac{\partial f_1}{\partial z_i} + \sum_{k=1}^m a^i_{1k}(z,s) f_k,\ \frac{\partial f_2}{\partial z_i} + \sum_{k=1}^m a^i_{2k}(z,s) f_k,\ \dots,\ \frac{\partial f_m}{\partial z_i} + \sum_{k=1}^m a^i_{mk}(z,s) f_k\Big),$$

$$P_1 f = T_n f,\quad P_j f = T_{n-j+1}(P_{j-1} f)\quad (j=2, 3, \cdots, n-1),$$

$$Pf = P_n f$$

for $f={}^t(f_1, f_2, \cdots, f_m) \in O^m$, where we denote the column vector f by ${}^t(f_1, f_2, \cdots, f_m)$ in order to conserve natural resources.

L. Ehrenpreis [2] considered an application of the sheaf theory to differential equations and gave a criterion for the existence of global solutions of differential equations where the existence of local solutions are assured. J. Kajiwara [5] applied the method of Ehrenpreis to ordinary differential equations in the analytic category. I. Wakabayashi [14] gave examples of domain of holomorphy in which an equation is not globally solvable. The equation $\frac{\partial f}{\partial z_1} = g$ is such an example, because there exists a simply connected domain in C^3 on which $\frac{\partial f}{\partial z_1} = g$

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has no global solution for some holomorphic functions g. H. Suzuki [12] stated a necessary and sufficient condition for the global existence of holomorphic solutions. S. I. Pinčuk [10] found sufficient conditions to solve the formulated problem and he found necessary and sufficient conditions for the solution of the problem. J. Kajiwara–T. Mori [6] obtained the necessary and sufficient condition that for any function $g\in H^0(\Omega, O^m)$ there is a function $f\in H^0(\Omega, O^m)$ satisfying the inhomogeneous equation $P_n f=g$ in case that $n=1$. M. Harita [4] obtained the condition in case that $n>1$. And J. Kajiwara–K. H. Shon [8] have obtained the equivalent relations for $H^1(\Omega, \operatorname{Ker} P_n)=0$ in case that $n=1$.

At first, we generalize the result of Kajiwara–Shon [8]. The method are based on the above [4, 8]. Nextly, we obtain the vanishing theorem of cohomology groups for domains, those are not Stein.

2. Preliminaries

Let D be a Stein domain of the product space $C\times S$, $\mathcal{O}$ be the sheaf of germs of holomorphic functions on D and $a_{pq}(z,s)$ be holomorphic functions on D. In case that $n=1$ of Section 1, we put $T_n f=Tf$ and let Ker T be the kernel of T. Let $\phi: D\longrightarrow S$ be the canonical projection. For $(z,s)\in D$, let $D(z,s)$ be the connected component of $\phi^{-1}(s)$ in $C\times\{s\}$ containing (z,s), $\tilde{D}$ be the set of all cuts $D(z,s)$ for all $(z,s)\in D$, $\tilde{D}_s$ be the set of all simply connected $D(z,s)$ for $(z,s)\in D$, and define the mapping $\phi: \tilde{D}\longrightarrow S$ by $\phi(D(z,s))=s\in S$.

THEOREM 2.1 ([8]). *If $\tilde{D}=\tilde{D}_s$, if there exists a domain E in $C\times S$ containing D such that all coefficients $a_{pq}(z,s)$ are holomorphic in E and that the space $\tilde{E}$ of cuts $E(z,s)$, $(z,s)\in E$, is a Hausdorff space and if the parameter space S is a Stein manifold, then the following properties* (1), (2) *and* (3) *are equivalent*:

(1) $H^1(D, \operatorname{Ker} T)=0$.

(2) *The dimension of $H^1(D, \operatorname{Ker} T)$ is finite or countably infinite.*

(3) *The set $\tilde{D}$ is a Stein manifold.*

THEOREM 2.2 ([8]). *Let D be a Stein domain of the product space $C\times S$ of C and a pure finite dimensional Stein space S. If $H^1(D, \operatorname{Ker} T)$*

$=\dfrac{H^0(D,\mathcal{D}^m)}{TH^0(D,\mathcal{D}^m)}=0$, *then either* $D(z,s)$ *is simultaneously simply connected for any* $(z,s)\in D$ *or* $D(z,s)$ *is simultaneously doubly connected and satisfies* $H^0(D(z,s),\ \mathrm{Ker}\ T)=0$ *for any* $(z,s)\in D$.

In case that $D(z,s)$ *is a doubly connected domain with* $H^0(Dz,s),\ \mathrm{Ker}\ T)=0$ *for any* $(z,s)\in D$, *then* $H^1(D,\ \mathrm{Ker}\ T)=0$ *holds if and only if* $\tilde{D}$ *is a Hausdorff space.*

In case that $D(z,s)$ *is a simply connected domain for any* $(z,s)\in D$, *if the dimension of* $H^1(D,\ \mathrm{Ker}\ T)$ *is finite or countably infinite and if all coefficients* $a_{pq}(z,s)$ *are holomorphic in a domain* E *of* $\mathbf{C}\times S$ *containing* D *such that* $\tilde{E}$ *is a Hausdorff space, then* $\tilde{D}$ *is a Hausdorff space and the domain* (D,ϕ) *over* S *is a domain of meromorphy. Moreover, if* S *is a Stein manifold then* $\tilde{D}$ *is also a Stein manifold. Conversely, if* $\tilde{D}=\tilde{D}_s$ *is a Stein space, then we have* $H^1(D,\ \mathrm{Ker}\ T)=0$.

3. Global holomorphic solutions

Let $\phi_i:\Omega\longrightarrow\{(z_1,\cdots,z_{i-1},\ z_{i+1},\cdots,z_n,s)\}$ be the canonical projection. For $(z,s)=(z_1,z_2,\cdots,z_n,s)\in\Omega$, let $\Omega_i(z,s)$ be the connected component of $\phi_i^{-1}(z_1,\cdots,z_{i-1},z_{i+1},\cdots,z_n,s)\cap\Omega$ in $\mathbf{C}^n\times\{s\}$ containing $(z,s)\in\Omega$ and $\tilde{\Omega}_i$ be the set of all cuts $\Omega_i(z,s)$ for any $(z,s)\in\Omega$ $(i=1,2,\cdots,n)$. We define the mapping $\phi_i:\tilde{\Omega}_i\longrightarrow\{(z_1,\cdots,z_{i-1},z_{i+1},\cdots,z_n,s)\}$ by $\phi_i(\Omega_i(z,s))=(z_1,\cdots,z_{i-1},z_{i+1},\cdots,z_n,s)$ for any $(z,s)\in\Omega$ and the canonical mapping $\eta_i:\Omega\longrightarrow\tilde{\Omega}_i$ by $\eta_i(z_1,z_2,\cdots,z_n,s)=\Omega_i(z,s)$ for any $(z,s)\in\Omega$ $(i=1,2,\cdots,n)$. We define in the space $\tilde{\Omega}_i$ the strongest topology so that the mapping η_i is continuous. Then the mapping ϕ_i is a local homeomorphism. We have short exact sequences of sheaves

$$0\longrightarrow\mathrm{Ker}\ T_i\longrightarrow\mathcal{O}^m\xrightarrow{T_i}\mathcal{O}^m\longrightarrow0,$$

$$0\longrightarrow\mathrm{Ker}\ P_j\longrightarrow\mathcal{O}^m\xrightarrow{P_j}\mathcal{O}^m\longrightarrow0$$

and long exact sequences of cohomology groups

$$\cdots\longrightarrow H^0(\Omega,\mathcal{O}^m)\xrightarrow{T_i}H^0(\Omega,\mathcal{O}^m)\longrightarrow H^1(\Omega,\ \mathrm{Ker}\ T_i)$$
$$\longrightarrow H^1(\Omega,\mathcal{O}^m)\longrightarrow H^1(\Omega,\mathcal{O}^m)\longrightarrow\cdots,$$

$$\cdots \longrightarrow H^0(\Omega, \mathcal{O}^m) \xrightarrow{P_j} H^0(\Omega, \mathcal{O}^m) \longrightarrow H^1(\Omega, \text{ Ker } P_j) \longrightarrow H^1(\Omega, \mathcal{O}^m) \longrightarrow H^1(\Omega, \mathcal{O}^m) \longrightarrow \cdots$$

for $i, j=1, 2, \cdots, n$. Since Ω is a Stein domain, we have $H^t(\Omega, \mathcal{O}^m)=0$ for $t \geqq 1$ and

$$H^1(\Omega, \text{ Ker } T_i)=H^0(\Omega, \mathcal{O}^m)/T_i H^0(\Omega, \mathcal{O}^m),$$
$$H^1(\Omega, \text{ Ker } P_j)=H^0(\Omega, \mathcal{O}^m)/P_j H^0(\Omega, \mathcal{O}^m).$$

A necessary and sufficient condition for $H^1(\Omega, \text{ Ker } P)=0$ is that every function which is locally of a form $Pf=g$ is also globally of the form (see [2]), and a necessary and sufficient condition that for any function $g \in H^0(\Omega, \mathcal{O}^m)$ there exists a function $f \in H^0(\Omega, \mathcal{O}^m)$ satisfying the form $Pf=g$ is that there holds $H^1(\Omega, \text{ Ker } P)=0$.

Let $E_i(z, s)$ be the connected component of $\psi_i^{-1}(z_1, \ldots, z_{i-1}, z_{i+1}, \cdots, z_n, s) \cap E_i$ in $C^n \times \{s\}$ containing $(z, s) \in E_i$ for any domain E_i in $C^n \times S$. Here after, we consider the case that $\Omega_i(z, s)$ is simply connected for each $(z, s) \in \Omega$ and $i=1, 2, \cdots, n$.

LEMMA 3.1. *Let $\Omega_1(z, s)$ be a simply connected domain for $(z, s) \in \Omega$. If the dimension of $H^1(\Omega, \text{ Ker } P)$ is finite or countably infinite, then the dimensions of $H^1(\Omega, \text{ Ker } T_1)$ and $H^1(\Omega, \text{ Ker } P_{n-1})$ are finite or countably infinite, respectively.*

Proof. Since the dimension of $H^1(\Omega, \text{ Ker } P)$ is finite or countably infinite, we have dim $H^1(\Omega, \text{ Ker } P) < +\infty$ by Y. T. Siu [11, Theorem 4] and then $H^1(\Omega, \text{ Ker } P)=0$ by Theorem 2.1. Therefore, we have $H^0(\Omega, \mathcal{O}^m)=PH^0(\Omega, \mathcal{O}^m)$. Then there exists a function $f \in H^0(\Omega, \mathcal{O}^m)$ such that $Pf=g$ for any function $g \in H^0(\Omega, \mathcal{O}^m)$. Letting $f^1=P_{n-1}f$, then we have $T_1 f^1=P_n f=g$. Hence we have $H^1(\Omega, \text{ Ker } T_1)=0$, that is, the dimension of $H^1(\Omega, \text{ Ker } T_1)$ is finite or countably infinite. For any function $g \in H^0(\Omega, \mathcal{O}^m)$, we have $T_1 g \in H^0(\Omega, \mathcal{O}^m)$. Hence there exists a function $h \in H^0(\Omega, \mathcal{O}^m)$ such that $T_1 g=Ph=T_1(P_{n-1}h)$ for any $g \in H^0(\Omega, \mathcal{O}^m)$. Thus we have $T_1(P_{n-1}f-g)=0$, and then $P_{n-1}f=g$ for any function $g \in H^0(\Omega, \mathcal{O}^m)$. So we have proved that $H^1(\Omega, \text{Ker } P_{n-1})=0$.

LEMMA 3.2. *Let $\Omega_i(z, s)$ be simply connected domains for all $1 \leqq i \leqq n$ and $(z, s) \in \Omega$. If the dimension of $H^1(\Omega, \text{ Ker } P)$ is finite or countably infinite, if there exists a domain E_i in $C^n \times S$ containing Ω for each $i=$*

$1, 2, \cdots, n$ such that all coefficients $a^i_{pq}(z, s)$ are holomorphic in E_i and if the space $\tilde{E}_i$ of cuts $E_i(z, s)$, $(z, s) \in E_i$, is a Hausdorff space for each $i=1, 2, \cdots, n$, then the dimension of $H^1(\Omega$, Ker $T_i)$ is finite or countably infinite, $\tilde{\Omega}_i$ is a Hausdorff space and the domain $(\tilde{\Omega}_i, \phi_i)$ over the Stein space S is a domain of meromorphy for each $i=1, 2, \cdots, n$.

Proof. For $i=1$, we have the result by Lemma 3.1 and Theorem 2.2. Suppose that the dimension of $H^1(\Omega$, Ker $P_k)$ is finite or countably infinite for $k<n$. By Lemma 3.1, we have the dimension of $H^1(\Omega$, Ker $P_{n-k})$ is finite or countably infinite and then $H^0(\Omega, O^m)=P_{n-k}H^0(\Omega, O^m)$. Since $T_{k+1}(P_{n-k-1}f)=P_{n-k}f$ for any $f \in H^0(\Omega, O^m)$, we have $H^1(\Omega$, Ker $T_{k+1})=0$. That is, the dimension of $H^1(\Omega$, Ker $T_{k+1})$ is finite or countably infinite. And the remainder statements are desired by Theorem 2.2.

THEOREM 3.3. *Let Ω be a Stein domain of $C^n \times S$ and $\Omega_i(z, s)$ be simply connected domains for $(z, s) \in \Omega$ and $1 \leqq i \leqq n$. If the dimension of H^{1} $(\Omega$, Ker $P)$ is finite or countably infinite, if there exists a domain E_i in $C^n \times S$ containing Ω for each $i=1, 2, \cdots, n$ such that all coefficients $a^i_{pq}(z, s)$ are holomorphic in E_i and if the space $\tilde{E}_i$ of cuts $E_i(z, s)$, $(z, s) \in E_i$, is a Hausdorff space for each $i=1, 2, \cdots, n$, then the dimension of $H^1(\Omega$, Ker $T_i)$ is finite or countably infinite, $\tilde{\Omega}_i$ is a Hausdorff space and the domain $(\tilde{\Omega}_i, \phi_i)$ over the Stein space S is a domain of meromorphy for each $i=1, 2, \cdots, n$. Conversely, if the simply connected domain $\Omega_i(z, s)$ is a Stein space for each $i=1, 2, \cdots, n$, then $H^1(\Omega$, Ker $P)=0$.*

Proof. By Theorem 2.2 and Lemma 3.1 and 3.2, we have the theorem.

K. Oka [9] proved that every domain over C^n analytically convex in the sence of Hartogs is a domain of holomorphy. Therefore a domain of meromorphy over C^n coincides eith a domain of holomorphy over C^n. J. Kajiwara–E. Sakai [7] proved that the envelope of meromorphy of a domain over a Stein manifold S with respect to a family of meromorphic function on the domain is p-convex in the sence of F. Docquier–H. Grauert [1] and, therefore, is a Stein manifold. Especially, a domain of meromorphy over S coincides with a domain of holomorphy over S.

LEMMA 3.4. *Under the assumption of Theorem* 3.3, *if* S *is a Stein manifold, then* $\tilde{\Omega}_i$ *are Stein manifolds for all* $i=1, 2, \cdots, n$.

Proof. For each $i=1, 2, \cdots, n$, since the unramified domain $(\tilde{\Omega}_i, \phi_i)$ over the Stein manifold S is a domain of meromorphy by Theorem 3.3, it is pseudoconvex by Kajiwara–Sakai [7]. Thus $\tilde{\Omega}_i$ are Stein manifolds for all $i=1, 2, \ldots, n$ by Docquier–Grauert [1].

In case that $U_1 \subset U_2 \subset \cdots$ be a sequence of open Stein subsets in X and $U=\bigcup_{j=1}^{\infty} U_j$, if X is a Stein manifold, it is known that U is Stein. And if X is a Stein space, it is not known whether U should be Stein. J. E. Fornaess [3] has given an example of a sequence of increasing Stein subsets in a manifold whose union is not Stein. If U_1 and U_2 are open Stein subsets of Stein space X, if $U=U_1 \cup U_2 \subset\subset X$ and if $\dim H^1(U, 0) < \infty$, then U is Stein by L. M. Tovar [13].

Let $\tilde{\Omega}_{ij}=\tilde{\Omega}_i \cup \tilde{\Omega}_j$ for the above Stein manifold $\tilde{\Omega}_i$, each i and j. Then the union $\tilde{\Omega}_{ij}$ is not necessarily Stein. We consider exclusively the case that $\tilde{\Omega}_i \cap \tilde{\Omega}_j \neq \phi$.

THEOREM 3.5. *Under the assumption of Lemma* 3.4, *if in addition* $\Omega \subset E_i \subset\subset C^n \times S$ *for each* $i=1, 2, \cdots, n$, *then*

$$H^1(\tilde{\Omega}_{jk}, \zeta) = H^0(\tilde{\Omega}_j \cap \tilde{\Omega}_k, \zeta) / R(H^0(\tilde{\Omega}_j, \zeta) \oplus H^0(\tilde{\Omega}_k, \zeta))$$

for some mapping R *and*

$$H^q(\tilde{\Omega}_{jk}, \zeta) = 0 \quad (q \geqq 2)$$

for any coherent analytic sheaf ζ *on* $C^n \times S$ $(j, k=1, 2, \cdots, n)$.

Proof. The intersection $\tilde{\Omega}_j \cap \tilde{\Omega}_k$ is a Stein manifold for each $j, k = 1, 2, \cdots, n$. For the coherent analytic sheaf ζ, we have the Mayer–Vietoris exact sequence

$$\begin{aligned} &0 \longrightarrow H^0(\tilde{\Omega}_{jk}, \zeta) \longrightarrow H^0(\tilde{\Omega}_j, \zeta) \oplus H^0(\tilde{\Omega}_k, \zeta) \\ &\xrightarrow{R} H^0(\tilde{\Omega}_j \cap \tilde{\Omega}_k, \zeta) \longrightarrow H^1(\tilde{\Omega}_{jk}, \zeta) \longrightarrow \cdots \\ &\longrightarrow H^{q-1}(\tilde{\Omega}_j \cap \tilde{\Omega}_k, \zeta) \longrightarrow H^q(\tilde{\Omega}_{jk}, \zeta) \longrightarrow \\ &H^q(\tilde{\Omega}_j, \zeta) \oplus H^q(\tilde{\Omega}_k, \zeta) \longrightarrow \cdots \end{aligned}$$

for each $j, k=1, 2, \cdots, n$. By Lemma 3.4 and the theorem B of Cartan, we have

$$H^q(\tilde{\Omega}_j, \zeta)=H^q(\tilde{\Omega}_k, \zeta)=0,$$
$$H^q(\tilde{\Omega}_j \cap \tilde{\Omega}_k, \zeta)=0 \quad (q \geqq 1)$$

for all $j, k=1, 2, \cdots, n$. Therefore, we have

$$H^1(\tilde{\Omega}_{jk}, \zeta)=H^0(\tilde{\Omega}_j \cap \tilde{\Omega}_k, \zeta)/R(H^0(\tilde{\Omega}_j, \zeta) \oplus H^0(\tilde{\Omega}_k, \zeta))$$

and

$$H^q(\tilde{\Omega}_{jk}, \zeta)=0 \quad (q \geqq 2)$$

for all $j, k=1, 2, \cdots, n$.

COROLLARY 3.6. *A necessary and sufficient condition that for any function* $g \in H^0(\tilde{\Omega}_j \cap \tilde{\Omega}_k, \zeta)$ *there exists a function* $f \in (H^0(\tilde{\Omega}_j, \zeta) \oplus H^0(\tilde{\Omega}_k, \zeta))$ *satisfying a form* $Rf=g$ *is that there holds* $H^1(\tilde{\Omega}_{jk}, \zeta)=0$. *And a necessary and sufficient condition for* $H^1(\tilde{\Omega}_{jk}, \zeta)=0$ *is that every function which is locally of the form* $Rf=g$ *is also globally of the form.*

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